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The Homogeneous and Isotropic Universe

Notation

In this book we denote the derivative with respect to physical time by a prime and the derivative with respect to conformal time by a dot,

$$\tau = \text{physical (cosmic) time} \qquad \frac{dX}{d\tau} \equiv X', \tag{1.1}$$

$$t = \text{conformal time} \qquad \frac{dX}{dt} \equiv \dot{X}. \tag{1.2}$$

Spatial 3-vectors are denoted by a boldface symbol such as $\mathbf{k}$ or $\mathbf{x}$ whereas four-dimensional spacetime vectors are denoted as $x = (x^\mu)$.

We use the metric signature $(-, +, +, +)$ throughout the book.

The Fourier transform is defined by

$$f(\mathbf{k}) = \int d^3x \, f(\mathbf{x}) \, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{1.3}$$

so that

$$f(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3k \, f(\mathbf{k}) \, e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{1.4}$$

We use the same letter for $f(\mathbf{x})$ and for its Fourier transform $f(\mathbf{k})$. The spectrum $P_f(k)$ of a statistically homogeneous and isotropic random variable f is given by

$$\langle f(\mathbf{k}) f^*(\mathbf{k}') \rangle = (2\pi)^3 \, \delta(\mathbf{k} - \mathbf{k}') \, P_f(k). \tag{1.5}$$

Since it is isotropic, $P_f(k)$ is a function only of the modulus $k = |\mathbf{k}|$.

Throughout this book we use units where the speed of light, c; Planck's constant, $\hbar$; and Boltzmann's constant, k_B, are unity: $c = \hbar = k_B = 1$. Length and time therefore have the same units and energy, mass, and momentum also have the same units, which are inverse to the unit of length. Temperature has the same units as energy. We may use cm^{-1} to measure energy, mass, and temperature, or eV^{-1} to measure distances or times. We shall use whatever unit is convenient to discuss a given problem. Conversion factors can be found in Appendix 1.

1.1 Homogeneity and Isotropy

Modern cosmology is based on the hypothesis that our Universe is to a good approximation homogeneous and isotropic on sufficiently large scales. This relatively bold assumption is often called the "cosmological principle." It is an extension of the Copernican principle stating that not only should our place in the Solar System not be a special one, but also that the position of the Milky Way in the Universe should be in no way statistically distinguishable from the position of other galaxies. Furthermore, no direction should be distinguished. The Universe looks statistically the same in all directions. This, together with the hypothesis that the matter density and geometry of the Universe are smooth functions of the position, implies homogeneity and isotropy on sufficiently large scales. Isotropy around each point together with analyticity actually already implies homogeneity of the Universe.[1] A formal proof of this quite intuitive result can be found in Straumann (1974).

But which scale is "sufficiently large"? Certainly not the Solar System or our Galaxy. But also not the size of galaxy clusters. [In cosmology, distances are usually measured in Mpc (Megaparsec). 1 Mpc $= 3.2615 \times 10^6$ light years $= 3.0856 \times 10^{24}$ cm is a typical distance between galaxies; the distance between our neighbor Andromeda and the Milky Way is about 0.7 Mpc. These and other connections between frequently used units can be found in Appendix 1.]

It turns out that the scale at which the *galaxy distribution* becomes homogeneous is difficult to determine. From the analysis of the Sloan Digital Sky Survey (SDSS) it has been concluded that the irregularities in the galaxy density are still on the level of a few percent on scales of 100 Mpc (Hogg *et al.*, 2005). Fortunately, we know that the *geometry* of the Universe shows only small deviations from the homogeneous and isotropic background, already on scales of a few Mpc. The geometry of the Universe can be tested with the peculiar motion of galaxies, with lensing, and in particular with the cosmic microwave background (CMB).

The small deviations from homogeneity and isotropy in the CMB are of utmost importance, since, most probably, they represent the "seeds," that, via gravitational instability, have led to the formation of large-scale structure, galaxies, and eventually solar systems with planets that support life in the Universe.

[1] If "analyticity" is not assumed, the matter distribution could also be fractal and still statistically isotropic around each point. For a detailed elaboration of this idea and its comparison with observations see Sylos Labini *et al.* (1998).

Furthermore, we suppose that the initial fluctuations needed to trigger the process of gravitational instability stem from tiny quantum fluctuations that have been amplified during a period of inflationary expansion of the Universe. I consider this connection of the microscopic quantum world with the largest scales of the Universe to be of breathtaking philosophical beauty.

In this chapter we investigate the background Universe. We shall first discuss the geometry of a homogeneous and isotropic spacetime. Then we investigate two important events in the thermal history of the Universe. Finally, we study the paradigm of inflation. This chapter lays the basis for the following ones where we shall investigate *fluctuations* on the background, most of which can be treated in first-order perturbation theory.

1.2 The Background Geometry of the Universe

1.2.1 The Friedmann Equations

In this section we assume a basic knowledge of general relativity. The notation and sign convention for the curvature tensor that we adopt are specified in Appendix 2, Section A2.1.

Our Universe is described by a four-dimensional spacetime $(\mathcal{M}, g)$ given by a pseudo-Riemannian manifold $\mathcal{M}$ with metric g. A homogeneous and isotropic spacetime is one that admits a slicing into homogeneous and isotropic, that is, maximally symmetric, 3-spaces. There is a preferred geodesic time coordinate τ, called "cosmic time," such that the 3-spaces of constant time, $\Sigma_\tau = \{\mathbf{x}|(\tau, \mathbf{x}) \in \mathcal{M}\}$, are maximally symmetric spaces, hence spaces of constant curvature. The metric g is therefore of the form

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -d\tau^2 + a^2(\tau)\gamma_{ij}\, dx^i\, dx^j. \tag{1.6}$$

The function $a(\tau)$ is called the scale factor and γ_{ij} is the metric of a 3-space of constant curvature K. Depending on the sign of K this space is locally isometric to a 3-sphere ($K > 0$); a three-dimensional pseudo-sphere ($K < 0$); or flat, Euclidean space ($K = 0$). In later chapters of this book we shall mainly use "conformal time" t defined by $a\, dt = d\tau$, so that

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = a^2(t)\left(-dt^2 + \gamma_{ij}\, dx^i\, dx^j\right). \tag{1.7}$$

The geometry and physics of homogeneous and isotropic solutions to Einstein's equations were first investigated mathematically in the early 1920s by Friedmann (1922, 1924) and physically as a description of the observed expanding Universe in 1927 by Lemaître.[2] Later, Robertson (1936), Walker (1936), and others rediscovered the Friedmann metric and studied several additional aspects. However, since we consider the contributions by Friedmann and Lemaître to be far more fundamental than the subsequent work, we shall call a homogeneous and isotropic solution to Einstein's equations a "Friedmann–Lemaître universe" (FL universe) in this book.

It is interesting to note that the Friedmann solution breaks Lorentz invariance. Friedmann universes are not invariant under boosts; there is a preferred cosmic time τ, the proper time of an observer who sees a spatially homogeneous and isotropic universe. Like so often in physics, the Lagrangian and therefore also the field equations of general relativity are invariant under Lorentz transformations, but a specific solution in general is not. In that sense we are back to Newton's vision of an absolute time. But on small scales, for example, the scale of a laboratory, this violation of Lorentz symmetry is, of course, negligible.

The topology is not determined by the metric and hence by Einstein's equations. There are many compact spaces of negative or vanishing curvature (e.g., the torus), but there are no infinite spaces with positive curvature. A beautiful treatment of the fascinating, but difficult, subject of the topology of spaces with constant curvature and their classification is given in Wolf (1974). Its applications to cosmology are found in Lachieze-Rey and Luminet (1995).

Forms of the metric γ, which we shall often use, are

$$\gamma_{ij}\, dx^i\, dx^j = \frac{\delta_{ij}\, dx^i\, dx^j}{(1 + \frac{1}{4} K\rho^2)^2}, \tag{1.8}$$

$$\gamma_{ij}\, dx^i\, dx^j = dr^2 + \chi^2(r)\left(d\theta^2 + \sin^2(\theta)\, d\varphi^2\right), \tag{1.9}$$

$$\gamma_{ij}\, dx^i\, dx^j = \frac{dR^2}{1 - KR^2} + R^2\left(d\theta^2 + \sin^2(\theta)\, d\varphi^2\right), \tag{1.10}$$

where in Eq. (1.8)

$$\rho^2 = \sum_{i,j=1}^{3} \delta_{ij} x^i x^j, \quad \text{and} \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else}, \end{cases} \tag{1.11}$$

and in Eq. (1.9);

$$\chi(r) = \begin{cases} r & \text{in the Euclidean case,} \quad K = 0, \\ \frac{1}{\sqrt{K}} \sin(\sqrt{K} r) & \text{in the spherical case,} \quad K > 0, \\ \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|} r) & \text{in the hyperbolic case,} \quad K < 0. \end{cases} \tag{1.12}$$

[2] In the English translation of (Lemaître, 1927) from 1931 Lemaître's somewhat premature but pioneering arguments that the observed Universe is actually expanding have been omitted.

Often one normalizes the scale factor such that $K = \pm 1$ whenever $K \neq 0$. One has, however, to keep in mind that in this case r and K become dimensionless and the scale factor a has the dimension of length. If $K = 0$ we can normalize a arbitrarily. We shall usually normalize the scale factor such that $a_0 = 1$ and the curvature is not dimensionless. The coordinate transformations that relate these coordinates are determined in Exercise 1.1.

Owing to the symmetry of spacetime, the energy–momentum tensor can only be of the form

$$(T_{\mu\nu}) = \begin{pmatrix} -\rho g_{00} & \mathbf{0} \\ \mathbf{0} & P g_{ij} \end{pmatrix}. \tag{1.13}$$

There is no additional assumption going into this ansatz, such as the matter content of the Universe being an ideal fluid. It is a simple consequence of homogeneity and isotropy and is also verified for scalar field matter, a viscous fluid, or free-streaming particles in a FL universe. As usual, the energy density ρ and the pressure P are defined as the time- and space-like eigenvalues of $(T_\nu^{\ \mu})$.

The Einstein tensor can be calculated from the definition (A2.12) and Eqs. (A2.32)–(A2.39),

$$G_{00} = 3\left[\left(\frac{a'}{a}\right)^2 + \frac{K}{a^2}\right] \quad \text{(cosmic time)}, \tag{1.14}$$

$$G_{ij} = -\left(2a''a + a'^2 + K\right)\gamma_{ij} \quad \text{(cosmic time)}, \tag{1.15}$$

$$G_{00} = 3\left[\left(\frac{\dot a}{a}\right)^2 + K\right] \quad \text{(conformal time)}, \tag{1.16}$$

$$G_{ij} = -\left(2\left(\frac{\dot a}{a}\right)^{\bullet} + \left(\frac{\dot a}{a}\right)^2 + K\right)\gamma_{ij} \quad \text{(conformal time)}. \tag{1.17}$$

The Einstein equations relate the Einstein tensor to the energy–momentum content of the Universe via $G_{\mu\nu} = 8\pi G T_{\mu\nu} - g_{\mu\nu}\Lambda$. Here Λ is the so-called cosmological constant. In an FL universe the Einstein equations become

$$\left(\frac{a'}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \quad \text{(cosmic time)}, \tag{1.18}$$

$$2\frac{a''}{a} + \frac{(a')^2}{a^2} + \frac{K}{a^2} = -8\pi G P + \Lambda \quad \text{(cosmic time)}, \tag{1.19}$$

$$\left(\frac{\dot a}{a}\right)^2 + K = \frac{8\pi G}{3}a^2\rho + \frac{a^2\Lambda}{3} \quad \text{(conformal time)}, \tag{1.20}$$

$$2\left(\frac{\dot a}{a}\right)^{\bullet} + \left(\frac{\dot a}{a}\right)^2 + K = -8\pi G a^2 P + a^2\Lambda \quad \text{(conformal time)}. \tag{1.21}$$

Energy "conservation," $T^{\mu\nu}_{;\mu} = 0$, yields

$$\dot{\rho} = -3(\rho + P)\left(\frac{\dot{a}}{a}\right) \quad \text{or, equivalently} \quad \rho' = -3(\rho + P)\left(\frac{a'}{a}\right). \tag{1.22}$$

This equation can also be obtained by differentiating Eq. (1.18) or (1.20) and inserting (1.19) or (1.21); it is a consequence of the contracted Bianchi identities (see Appendix 2, Section A2.1). Equations (1.18)–(1.21) are the Friedmann equations. The quantity

$$H(\tau) \equiv \frac{a'}{a} = \frac{\dot{a}}{a^2} \equiv \mathcal{H}a^{-1}, \tag{1.23}$$

is called the Hubble rate or the Hubble parameter, where $\mathcal{H}$ is the comoving Hubble parameter. At present, the Universe is expanding, so that $H_0 > 0$. We parameterize it by

$$H_0 = 100\, h \text{ km s}^{-1}\text{ Mpc}^{-1} \simeq 3.241 \times 10^{-18}\, h \text{ s}^{-1} \simeq 0.3336 \times 10^{-3} \text{ h Mpc}^{-1}.$$

Observations show (Freedman *et al.*, 2001) that $h \simeq 0.72 \pm 0.1$. Equation (1.22) is easily solved in the case $w = P/\rho = $ constant. Then one finds

$$\rho = \rho_0 (a_0/a)^{3(1+w)}, \tag{1.24}$$

where ρ_0 and a_0 denote the value of the energy density and the scale factor at present time, τ_0. In this book cosmological quantities indexed by a "0" are evaluated today, $X_0 = X(\tau_0)$. For nonrelativistic matter, $P_m = 0$, we therefore have $\rho_m \propto a^{-3}$ while for radiation (or any kind of massless particles) $P_r = \rho_r/3$ and hence $\rho_r \propto a^{-4}$. A cosmological constant corresponds to $P_\Lambda = -\rho_\Lambda$ and we obtain, as expected, $\rho_\Lambda = $ constant. If the curvature K can be neglected and the energy density is dominated by one component with $w = $ constant, inserting Eq. (1.24) into the Friedmann equations yields the solutions

$$a \propto \tau^{2/3(1+w)} \propto t^{2/(1+3w)} \qquad w = \text{constant} \neq -1, \tag{1.25}$$

$$a \propto \tau^{2/3} \propto t^2 \qquad w = 0, \quad \text{(dust)}, \tag{1.26}$$

$$a \propto \tau^{1/2} \propto t \qquad w = 1/3, \quad \text{(radiation)}, \tag{1.27}$$

$$a \propto \exp(H\tau) \propto 1/|t| \qquad w = -1, \quad \text{(cosmol. const.)}. \tag{1.28}$$

It is interesting to note that if $w < -1$, so-called phantom matter, we have to choose $\tau < 0$ to obtain an expanding universe and the scale factor diverges in finite time, at $\tau = 0$. This is the so-called big rip. Phantom matter has many problems but it is discussed in connection with the supernova type 1a (SN1a) data, which are compatible with an equation of state with $w < -1$ or with an ordinary cosmological constant (Caldwell *et al.*, 2003). For $w < -\frac{1}{3}$ the time coordinate t

has to be chosen as negative for the Universe to expand and spacetime cannot be continued beyond $t = 0$. But $t = 0$ corresponds to a cosmic time, the proper time of a static observer, $\tau = \infty$; this is not a singularity. (The geodesics can be continued until affine parameter ∞.)

We also introduce the adiabatic sound speed c_s determined by

$$c_s^2 = \frac{P'}{\rho'} = \frac{\dot{P}}{\dot{\rho}}. \tag{1.29}$$

From this definition and Eq. (1.22) it is easy to see that

$$\dot{w} = 3\mathcal{H}(1 + w)\left(w - c_s^2\right). \tag{1.30}$$

Hence w = constant if and only if $w = c_s^2$ or $w = -1$. Note that already in a simple mixture of matter and radiation $w \neq c_s^2 \neq$ constant (see Exercise 1.3).

Equation (1.18) implies that for a critical value of the energy density given by

$$\rho(\tau) = \rho_c(\tau) = \frac{3H^2}{8\pi G} \tag{1.31}$$

the curvature and the cosmological constant vanish. The value ρ_c is called the critical density. The ratio $\Omega_X = \rho_X/\rho_c$ is the "density parameter" of the component X. It indicates the fraction that the component X contributes to the expansion of the Universe. We shall make use especially of

$$\Omega_r \equiv \Omega_r(\tau_0) = \frac{\rho_r(\tau_0)}{\rho_c(\tau_0)}, \tag{1.32}$$

$$\Omega_m \equiv \Omega_m(\tau_0) = \frac{\rho_m(\tau_0)}{\rho_c(\tau_0)}, \tag{1.33}$$

$$\Omega_K \equiv \Omega_K(\tau_0) = \frac{-K}{a_0^2 H_0^2}, \tag{1.34}$$

$$\Omega_\Lambda \equiv \Omega_\Lambda(\tau_0) = \frac{\Lambda}{3H_0^2}. \tag{1.35}$$

1.2.2 The "Big Bang" and "Big Crunch" Singularities

We can absorb the cosmological constant into the energy density and pressure by redefining

$$\rho_{\text{eff}} = \rho + \frac{\Lambda}{8\pi G}, \qquad P_{\text{eff}} = P - \frac{\Lambda}{8\pi G}.$$

Since Λ is a constant and $\rho_{\text{eff}} + P_{\text{eff}} = \rho + P$, the conservation equation (1.22) still holds. A first interesting consequence of the Friedmann equations is obtained when subtracting Eq. (1.18) from (1.19). This yields

$$\frac{a''}{a} = -\frac{4\pi G}{3}(\rho_{\text{eff}} + 3P_{\text{eff}}). \tag{1.36}$$

Hence if $\rho_{\text{eff}} + 3P_{\text{eff}} > 0$, the Universe is decelerating. Furthermore, Eqs. (1.22) and (1.36) then imply that in an expanding and decelerating universe

$$\frac{\rho'_{\text{eff}}}{\rho_{\text{eff}}} < -2\frac{a'}{a},$$

so that ρ decays faster than $1/a^2$. If the curvature is positive, $K > 0$, this implies that at some time in the future, τ_{max}, the density has dropped down to the value of the curvature term, $K/a^2(\tau_{\text{max}}) = 8\pi G\rho_{\text{eff}}(\tau_{\text{max}})$. Then the Universe stops expanding and recollapses. Furthermore, this is independent of curvature; as a' decreases the curve $a(\tau)$ is concave and thus cuts the $a = 0$ line at some finite time in the past. This moment of time is called the "big bang." The spatial metric vanishes at this value of τ, which we usually choose to be $\tau = 0$; and spacetime cannot be continued to earlier times. This is not a coordinate singularity. From the Ricci tensor given in Eqs. (A2.32) and (A2.33) one obtains the Riemann scalar

$$R = 6\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2 + \frac{K}{a^2}\right],$$

which also diverges if $a \to 0$. Also the energy density, which grows faster than $1/a^2$ as $a \to 0$, diverges at the big bang.

If the curvature K is positive, the Universe contracts after $\tau = \tau_{\text{max}}$ and, since the graph $a(\tau)$ is convex, reaches $a = 0$ at some finite time τ_c, the time of the "big crunch." The big crunch is also a physical singularity beyond which spacetime cannot be continued.

It is important to note that this behavior of the scale factor can be implied only if the so-called strong energy condition holds, $\rho_{\text{eff}} + 3P_{\text{eff}} > 0$. This is illustrated in Fig. 1.1.

1.2.3 Cosmological Distance Measures

It is notoriously difficult to measure distances in the Universe. The position of an object in the sky gives us its angular coordinates, but how far away is the object from us? This problem had plagued cosmology for centuries. It took until 1915–1920 when Hubble discovered that the "spiral nebulae" are actually not situated inside our own galaxy but much further away. This then led to the discovery of the expansion of the Universe.

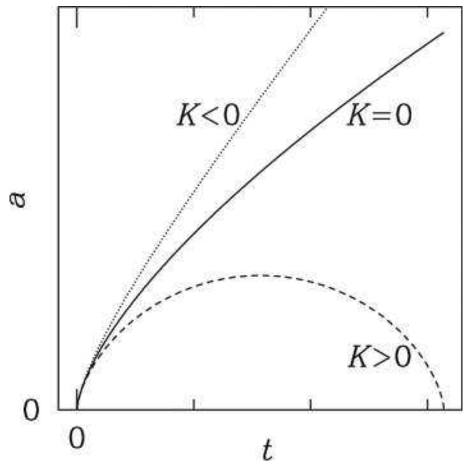

Fig. 1.1 The kinematics of the scale factor in a Friedmann–Lemaître universe that satisfies the strong energy condition, $\rho_{\rm eff} + 3P_{\rm eff} > 0$.

For cosmologically distant objects, a third coordinate, which is today relatively easy to obtain, is the redshift z experienced by the photons emitted from the object. A given spectral line with intrinsic wavelength λ is redshifted due to the expansion of the Universe. If it is emitted at some time τ, it reaches us today with wavelength $\lambda_0 = \lambda a_0/a(\tau) = (1+z)\lambda$. This leads to the definition of the cosmic redshift

$$z(\tau) + 1 = \frac{a_0}{a(\tau)}. \qquad (1.37)$$

On the other hand, an object at physical distance $d = a_0 r$ away from us, at redshift $z \ll 1$, recedes with speed $v = H_0 d$. To the lowest order in z, we have $\tau_0 - \tau \approx d$ and $a_0 \approx a(\tau) + a'(\tau_0 - \tau)$, so that

$$1 + z \approx 1 + \frac{a'}{a}(\tau_0 - \tau) \approx 1 + H_0 d.$$

For objects that are sufficiently close, $z \ll 1$. We therefore have $v \approx z$ and hence $H_0 = z/d$. This is the method usually applied to measure the Hubble constant.

There are different ways to measure distances in cosmology, all of which give the same result in a Minkowski universe but differ in an expanding universe. They are, however, simply related, as we shall see.

One possibility is to define the distance d_A to a certain object of given physical size Δ seen at redshift z_1 such that the angle subtended by the object is given by

$$\vartheta = \Delta/d_A, \qquad d_A = \Delta/\vartheta. \qquad (1.38)$$

This is the angular diameter distance; see Fig. 1.2.

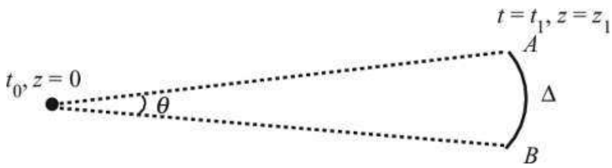

Fig. 1.2 The two ends of the object emit a flash simultaneously from A and B at z_1 which reaches us today. The angular diameter distance to A (or B) is defined by $d_A = \Delta/\vartheta$.

We now derive the expression

$$d_A(z) = \frac{1}{\sqrt{|\Omega_K|}H_0(1+z)}\chi\left(\sqrt{|\Omega_K|}H_0\int_0^z \frac{dz'}{H(z')}\right), \qquad (1.39)$$

for the angular diameter distance to redshift z. In a given cosmological model, this allows us to express the angular diameter distance for a given redshift as a function of the cosmological parameters.

To derive Eq. (1.39) we use the coordinates introduced in Eq. (1.9). Without loss of generality we set $r = 0$ at our position. We consider an object of physical size Δ at redshift z_1 simultaneously emitting a flash at both of its ends, A and B. Hence $r = r_1 = t_0 - t_1$ at the position of the flashes, A and B at redshift z_1. If Δ denotes the physical arc length between A and B we have $\Delta = a(t_1)\chi(r_1)\vartheta = a(t_1)\chi(t_0 - t_1)\vartheta$, that is,

$$\vartheta = \frac{\Delta}{a(t_1)\chi(t_0 - t_1)}. \qquad (1.40)$$

According to Eq. (1.38) the angular diameter distance to t_1 or z_1 is therefore given by

$$a(t_1)\chi(t_0 - t_1) \equiv d_A(z_1). \qquad (1.41)$$

To obtain an expression for $d_A(z)$ in terms of the cosmic density parameters and the redshift, we have to calculate $(t_0 - t_1)(z_1)$.

Note that in the case $K = 0$ we can normalize the scale factor a as we want, and it is convenient to choose $a_0 = 1$, so that comoving scales become physical scales today. However, for $K \neq 0$, we have already normalized a such that $K = \pm 1$ and $\chi(r) = \sin r$ or $\sinh r$. In this case, we have no normalization constant left and a_0 has the dimension of a length. The present spatial curvature of the Universe then is $\pm 1/a_0^2$.

The Friedmann equation Eq. (1.20) reads

$$\dot{a}^2 = \frac{8\pi G}{3}a^4\rho + \frac{1}{3}\Lambda a^4 - Ka^2, \qquad (1.42)$$