

The Mathematics of Finite Networks

Since the early eighteenth century, the theory of networks and graphs has matured into an indispensable tool for describing countless real-world phenomena. However, the study of large-scale features of a network often requires unrealistic limits, such as taking the network size to infinity or assuming a continuum. These asymptotic and analytic approaches can significantly diverge from real or simulated networks when applied at the finite scales of real-world applications. This book offers an approach to overcoming these limitations by introducing operator graph theory, an exact, non-asymptotic set of tools combining graph theory with operator calculus. The book is intended for mathematicians, physicists, and other scientists interested in discrete finite systems and their graph-theoretical description, and in delineating the abstract algebraic structures that characterise such systems. All the necessary background on graph theory and operator calculus is included for readers to understand the potential applications of operator graph theory.

MICHAEL RUDOLPH is a mathematical physicist of the French National Centre for Scientific Research at the Denis Poisson Institute. His research includes graph theory and classical number theory, and is directed towards understanding physical reality from an inherently finite discrete perspective, both mathematically and philosophically.

Cambridge University Press & Assessment
978-1-107-13443-0 — The Mathematics of Finite Networks
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The Mathematics of Finite Networks

An Introduction to Operator Graph Theory

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CAMBRIDGE UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India
103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.
It furthers the University's mission by disseminating knowledge in the pursuit of
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www.cambridge.org
Information on this title: www.cambridge.org/9781107134430
DOI: 10.1017/9781316466919

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First published 2022

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data

Names: Rudolph, Michael, 1969– author.

Title: The mathematics of finite networks : an introduction to operator
graph theory / Michael Rudolph.

Description: Cambridge ; New York, NY : Cambridge University Press, 2022. |
Includes bibliographical references and index.

Identifiers: LCCN 2021058077 (print) | LCCN 2021058078 (ebook) |

ISBN 9781107134430 (hardback) | ISBN 9781316466919 (epub)

Subjects: LCSH: Graph theory. | Operator theory. | System analysis. |

BISAC: COMPUTERS / General

Classification: LCC QA166 .R83 2022 (print) | LCC QA166 (ebook) |

DDC 511/.5–dc23/eng/20220126

LC record available at <https://lcn.loc.gov/2021058077>

LC ebook record available at <https://lcn.loc.gov/2021058078>

ISBN 978-1-107-13443-0 Hardback

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For Uesat
— *in memoriam* —

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Preface

Imagine for a moment that you stroll through a beautiful city like London with its dizzying array of inspiring detail. An overwhelming number of small streets, unpredictably intersecting with one another, wind around busy alleyways which themselves appear to stretch endlessly into all directions. Finding your way in this arguably chaotic mesh of interconnected paths and roadways by foot or by car without resorting to a memorised representation or detailed map of its layout would require the often time-consuming search through a vast number of possibilities until, eventually, the desired destination is reached. Taking a bird's-eye perspective, however, this confusing maze of small streets and mews seems to magically disappear and reveal, with concentric rings of expressways framing a star of boulevards which emanate from the metropole's centre, a different, much simpler pattern. Although such a coarser structural view certainly helps to confine your search, it does so at the expense of abstracting from finer details, – the small, narrow streets which are not only an integral part of the cultural and historical lifeblood of a city like London but reside at the very heart of its flair. Naturally, you might then wonder whether it is possible to hold on to these pertinent details alongside the more crude structural peculiarities which emerge at larger scales and hold, not unlike a skeleton, those details together.

When thinking of city maps, we instinctively envision a network of links along which an ever-changing flow of traffic is carried. Such an idealised description, however, is not limited to the maps we are all familiar with. From the interactions between atoms and subatomic particles to the gravitational forces which act between the billions of galaxies stretching across the known universe, from the transmission of electrical signals in our brains to the complexity of social interactions between people, most if not all of the phenomena we encounter, consciously or not, all around us find a natural representation in the form of networks. Indeed, it can be indubitably argued that the abstract

notion of interacting objects resides at the very heart of our conceptual understanding of nature as it touches upon the very fabric of physical reality with its finite and discrete makeup. It is for that reason that we now find the mathematical study of interconnected objects, the theory of networks and graphs, branching deeply through all fields of science and its innumerable applications.

Since its formal inception in the early eighteenth century, graph theory has constructed a solid foundation on which its conceptual and mathematical roots now comfortably rest. In doing so, it has amassed a plethora of useful tools which allow for a thorough qualification and quantification of the sheer uncountable number of networks we find all around us. Unfortunately, such a realisation of graph-theoretical concepts does not come without its limitations. All too often, it is the very nature of a studied system which also can, and does, throw stones in the way of its graph-theoretical formalisation or viable qualifying generalisations. Worse even, many real-world systems defiantly refuse an analytical treatment with the toolkit of graph theory and can only be studied numerically, thus, they are inadvertently yet inescapably tied to the efficiency of our computational hardware, or the lack thereof.

To bypass these limitations, and to capture the essential structural properties of real-world networks, the construction of simplified mathematical models is required. However, as in the example of London's map, many of the current mathematical approaches provide only a bird's-eye view, at the expense of the finer details, in order to reveal the large-scale features of the network that may exist, or to allow certain simplifications to become effective. Indeed, these approaches regularly consider unrealistic limits, such as taking the network size to infinity or abstracting from a network's defining discrete makeup by assuming a continuum. While such 'asymptotic' and 'analytic' approaches can certainly provide simple representations of the dominant structural features and, thus, often succeed at describing network structure in the aggregate, however, the equations at their core can diverge from real or simulated networks quite significantly, especially when applied to the finite scales occurring in the real world. What if the algorithms behind network models – and, for good models, behind real-world data as well – could be cast in mathematical terms that would allow for direct manipulations and, in some cases, exact solutions without the necessity of imposing limits? What if such algorithms could be cast into a form which would allow one to consider specific graph ensembles all at once, without the need to computationally construct and assess concrete individual realisation?

In this book, we aim to introduce, or at least propose, exactly such an approach to graph theory. The mathematical language used for formalising this discrete and finite algebraic framework lends itself to operator calculus,

and we have termed this framework *operator graph theory*. Irrespective of being tackled from a computational or more mathematical perspective, the investigation of large yet finite networks and their defining structural peculiarities still remains largely out of reach. Algorithmic approaches often become lost in a vast set of possibilities, as computational methods cannot cope with the combinatorial complexities inherent to many graph-theoretical problems; while asymptotic approaches are often unable to capture the very structure they are obliged to exclude, especially where real-world data are concerned. With this in mind, it is our great hope that operator graph theory may provide a viable language capable of casting such descriptions, by providing an exact, non-asymptotic set of tools which can be applied to, and provides links between, real-world data and their graph-theoretical representations.

This book is intended primarily for physicists, mathematicians, biologists and computer scientists, among others, who are interested in discrete, finite real-world systems that can be described in graph-theoretical terms, as well as in delineating the abstract algebraic structures characterising such systems. Including all the necessary background material on graph theory and operator calculus, and requiring no specific prerequisites other than linear algebra and calculus at the undergraduate level, this book is essentially self-contained and suitable for advanced undergraduates, postgraduates and researchers from a variety of backgrounds. However, it is important to stress that this book has to be understood as nothing more than a mere proposition which attempts the natural, yet arguably novel, fusion of graph theory and operator calculus. With many details in need of being fleshed out by the unforgiving demands of mathematical rigour still at large, I nevertheless hope to provide sufficient inspiration and motivation for an exciting explorative journey across uncharted waters. Let us take this journey together and see where it will lead us, because, after all, this is what science is all about.

I humbly admit that this book has been some years in the making, during a time of great professional and personal hardship. For that reason, I must first and foremost thank my ever-patient editor Lauren Cowles and her editorial assistant Amy He for their unwavering support and understanding nature. The same applies to my fellow colleagues at the Institute Denis Poisson in Tours and Orleans, who gave me in the ‘Year of Crazy’ a new comfortable home and awoke a memory of still-existing places in which only the sky is the limit, a cherished memory I thought had long been lost. When thinking about the tranquility of a new home, it is impossible not to see Madame Gigi and Daniel ‘Rambo72’ who not only helped me to build, from nothing, a secluded Asgardian paradise deep in the intoxicating beauty of Sarthe, France, but gave me

through their kindness, friendship and unconditional support the strength to master the uncompromising waves of fate crashing down on me. Writing these lines would also have been impossible without my ever-growing family of cats who finally can, like their ‘Old Man’, enjoy a simple life in nature and the freedom we are born with. Accompanied by the words of the great Charles Dickens, ‘what a greater gift than the love of a cat’, my beyond-infinite thanks go to Uesat and Black Mountain, Schmusie and Flusie, as well as the many cats I could not save from the abyss of human savagery. Last but not least, my profound and eternal thanks to Morgan and the two ‘Hepburns’, Audrey and Katherine, and Mark and Andrew, for their selfless support in the dark times I endured, as well as to Claude for keeping alive the fire of passion which burns in each true explorer of the mysteries of nature. To all of you I owe my life, and the candle of hope which, once again, is shining brightly deep inside me. Finally, thanks go to Lyle E. Muller for his help in preparing the original proposal for this book, as well as to J.A.G. Willow, C.O. Cain, S. Hower and L.S. Dee for their invaluable guidance, without which the conclusion of this work would have been impossible.