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Probability on Real Lie Algebras

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To Christine Jing and Sophie Wanlu.

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Notation

$\mathbb{K} = \mathbb{R}$, *resp.* $\mathbb{K} = \mathbb{C}$, denote the fields of real, *resp.* complex, numbers.

$\mathcal{B}(\mathbb{R})$ denotes the *Borel σ-algebra*, *i.e.*, the σ-algebra generated by the open subsets of $\mathbb{R}$.

$\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

$i = \sqrt{-1}$ denotes the complex square root of -1.

$\Im(z)$ denotes the imaginary part of $z \in \mathbb{C}$.

$\Re(z)$ denotes the real part of $z \in \mathbb{C}$.

$\operatorname{sgn} x \in \{-1, 0, 1\}$ denotes the sign of $x \in \mathbb{R}$.

$\delta_{n,m} = \mathbf{1}_{\{n=m\}} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}$ is the Kronecker symbol.

ϵ_x denotes the Dirac measure at the point x.

ℓ^2 denotes the space of complex-valued square-summable sequences.

$\mathfrak{h}$ denotes a (complex) separable Hilbert space and $\mathfrak{h}_{\mathbb{C}}$ denotes its complexification when $\mathfrak{h}$ is real.

"$\circ$" denotes the symmetric tensor product in Hilbert spaces.

$\Gamma_s(\mathfrak{h})$ denotes the symmetric Fock space over the real (*resp.* complexified) Hilbert space $\mathfrak{h}$ (*resp.* $\mathfrak{h}_{\mathbb{C}}$).

$B(\mathfrak{h})$ denotes the algebra of bounded operators over a Hilbert space $\mathfrak{h}$.

$\operatorname{tr} \rho$ denotes the trace of the operator ρ.

$|X|$ denotes the absolute value of a normal operator X, with $|X| := (X^*X)^{1/2}$ when X is not normal.

$|\phi\rangle\langle\psi|$ with $\phi, \psi \in \mathfrak{h}$ denotes the rank one operator on the Hilbert space $\mathfrak{h}$, defined by $|\phi\rangle\langle\psi|(v) = \langle\psi, v\rangle\phi$ for $v \in \mathfrak{h}$.

$[\cdot, \cdot]$ denotes the commutator $[X, Y] = XY - YX$.

$\{\cdot, \cdot\}$ denotes the anti-commutator $\{X, Y\} = XY + YX$.

Ad, *resp.* $\mathrm{ad}\, X$, denote the adjoint action on a Lie group, *resp.* Lie algebra.

$\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decreasing smooth functions.

$\mathcal{C}_0(\mathbb{R})$ denotes the set of continuous functions on $\mathbb{R}$, vanishing at infinity.

$\mathcal{C}_b^\infty(\mathbb{R})$ denotes the set of infinitely differentiable functions on $\mathbb{R}$ which are bounded together with all their derivatives.

$H^{p,\kappa}(\mathbb{R}^2)$ denotes the Sobolev space of orders $\kappa \in \mathbb{N}$ and $p \in [2, \infty]$.

$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$ denotes the standard gamma function.

$J_m(x)$ denotes the Bessel function of the first kind of order $m \geq 0$.

Preface

This monograph develops a pedagogical approach to the role of noncommutativity in probability theory, starting in the first chapter at a level suitable for graduate and advanced undergraduate students. The contents also aim at being relevant to the physics student and to the algebraist interested in connections with probability and statistics.

Our presentation of noncommutativity in probability revolves around concrete examples of relations between algebraic structures and probability distribution, especially via recursive relations among moments and their generating functions. In this way, basic Lie algebras such as the Heisenberg–Weyl algebra $\mathfrak{hw}$, the oscillator algebra $\mathfrak{osc}$, the special linear algebra $sl(2,\mathbb{R})$, and other Lie algebras such as $so(2)$ and $so(3)$, can be connected with classical probability distributions, notably the Gaussian, Poisson, and gamma distributions, as well as some other infinitely divisible distributions.

Based on this framework, the Chapters 1–3 allow the reader to directly manipulate examples and as such they remain accessible to advanced undergraduates seeking an introduction to noncommutative probability. This setting also allows the reader to become familiar with more advanced topics, including the notion of couples of noncommutative random variables via the use of Wigner densities, in relation with quantum optics.

The following chapters are more advanced in nature, and are targeted to the graduate and research levels. They include the results of recent research on quantum Lévy processes and the noncommutative Malliavin [75] calculus. The Malliavin calculus is introduced in both the commutative and noncommutative settings and contributes to a better understanding of the smoothness properties of Wigner densities.

While this text is predominantly based on research literature, part of the material has been developed for teaching in the course "Special topics in statistics" at the Nanyang Technological University, Singapore, in the second semester of academic year 2013–2014. We thank the students and participants for useful questions and suggestions.

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Uwe Franz
Nicolas Privault

Introduction

> Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field.
>
> *(P.A.M. Dirac, in* The Principles of Quantum Mechanics*.)*

Quantum probability addresses the challenge of merging the apparently distinct domains of algebra and probability, in view of physical applications. Those fields typically involve radically different types of thinking, which are not often mastered simultaneously. Indeed, the framework of algebra is often abstract and noncommutative while probability addresses the "fluctuating" but classical notion of a random variable, and requires a good amount of statistical intuition. On the other hand, those two fields combined yield natural applications, for *e.g.*, quantum mechanics. On a more general level, the noncommutativity of operations is a common real life phenomenon which can be connected to classical probability via quantum mechanics. Algebraic approaches to probability also have applications in theoretical computer science, cf. *e.g.*, [39].

In the framework of this noncommutative (or algebraic) approach to probability, often referred to as quantum probability, real-valued random variables on a classical probability space become special examples of noncommutative (or quantum) random variables. For this, a real-valued random variable on a probability space $(\Omega, \mathcal{F}, P)$ is viewed as (unbounded) self-adjoint multiplication operators acting on the Hilbert space $L^2(\Omega, P)$. This has led to the suggestion by several authors to develop a theory of quantum probability within the framework of operators and group representations in a Hilbert space, cf. [87] and references therein.

In this monograph, our approach is to focus on the links between the commutation relations within a given noncommutative algebra $\mathcal{A}$ on the one hand, and the combinatorics of the moments of a given probability distribution on the other hand. This approach is exemplified in Chapters 1–3. In this respect our point of view is consistent with the description of quantum probability by P.A. Meyer in [80] as a *set of prescriptions to extract probability from algebra*, based on various choices for the algebra $\mathcal{A}$.

For example, it is a well-known fact that the Gaussian distribution arises from the Heisenberg–Weyl algebra which is generated by three elements $\{P, Q, I\}$ linked by the commutation relation

$$[P, Q] = PQ - QP = 2iI.$$

It turns out similarly that other infinitely divisible distributions such as the gamma and continuous binomial distributions can be constructed via noncommutative random variables using representations of the special linear algebra $\mathfrak{sl}_2(\mathbb{R})$, or more simply on the affine algebra viewed as a sub-algebra of $\mathfrak{sl}_2(\mathbb{R})$. Other (joint) probability laws can be deduced in this setting; *e.g.*, one can construct noncommutative couples of random variables with gamma and continuous binomial marginals. Similarly, the Poisson distribution can be obtained in relation with the oscillator algebra. In Chapters 4 and 6, those basic examples are revisited and extended in the more general framework of quantum random variables on real Lie algebras. We often work on real Lie algebras given by complex Lie algebras with an involution, because calculations are more convenient on complexifications. The real Lie algebras can then be recovered as real subspaces of anti-Hermitian elements.

Since the elements of a Lie algebra $\mathfrak{g}$ can be regarded as functions on its dual $\mathfrak{g}^*$ it might be more precise to view a random variable $j : \mathfrak{g} \to \mathcal{A}$ as taking values in $\mathfrak{g}^*$. In this sense, this book deals with "probability on duals of real Lie algebras", which would better reflect the implicit dualisation in the definition of quantum probability spaces and quantum random variables. For simplicity of exposition we nonetheless prefer to work with the less precise terminology "probability on real Lie algebras". We refer to [10] and the references therein for further discussion and motivation of "noncommutative (or quantum) mathematics".

The notion of *joint distribution* for random vectors is of capital importance in classical probability theory. It also has an analog for couples of noncommutative random variables, through the definition of the (not necessarily positive) *Wigner* [124] density functions. In Chapter 7 we present a construction of joint densities for noncommutative random variables, based on functional calculus on real Lie algebras using the general framework of [7] and [8], in particular on the affine algebra. In that sense our presentation is also connected to the framework of standard quantum mechanics and quantum optics, where Wigner densities have various applications in, *e.g.*, time-frequency analysis, see, *e.g.*, the references given in [29] and [7].

Overall, this monograph puts more emphasis on noncommutative "problems with fixed time" as compared with "problems in moving time"; see, *e.g.*, [31] and [32] for a related organisation of topics in classical probability and stochastic calculus. Nevertheless, we also include a discussion of noncommutative stochastic processes via quantum Lévy processes in Chapter 8. Lévy processes, or stochastic processes with independent and stationary increments, are used as models for random fluctuations, *e.g.*, in physics and finance. In quantum physics the so-called quantum noises or quantum Lévy processes occur, *e.g.*, in the description of quantum systems coupled to a heat bath [47] or in the theory of continuous measurement [53]. See also [122] for a model motivated by lasers, and [2, 106] for the theory of Lévy processes on involutive bialgebras. Those contributions extend, in a sense, the theory of factorisable representations of current groups and current algebras as well as the theory of classical Lévy processes with values in Euclidean space or, more generally, semigroups. For a historical survey on the theory of factorisable representions and its relation to quantum stochastic calculus, see [109, section 5]. In addition, many interesting classical stochastic processes can be shown to arise as components of quantum Lévy processes, cf. *e.g.*, [1, 18, 42, 105].

We also intend to connect noncommutative probability with the Malliavin calculus, which was originally designed by P. Malliavin, cf. [75], as a tool to provide sufficient conditions for the smoothness of partial differential equation solutions using probabilistic arguments, see Chapter 9 for a review of its construction. Over the years, the Malliavin calculus has developed into many directions, including anticipating stochastic calculus and extensions of stochastic calculus to fractional Brownian motion, cf. [84] and references therein.

The Girsanov theorem is an important tool in stochastic analysis and the Malliavin calculus, and we derive its noncommutative, or algebraic version in Chapter 10, starting with the case of noncommutative Gaussian processes. By differentiation, Girsanov-type identities can be used to derive integration by parts formulas for the Wigner densities associated to the noncommutative processes, by following Bismut's argument, cf. [22]. In Chapter 10 we will demonstrate on several examples how quasi-invariance formulas can be obtained in such a situation. This includes the Girsanov formula for Brownian motion, as well as a quasi-invariance result of the gamma processes [111, 112], which actually appeared first in the context of factorisable representations of current groups [114], and a quasi-invariance formula for the Meixner process.

In Chapter 11 we present the construction of noncommutative Malliavin calculus on the Heisenberg–Weyl algebra [43], [44], which generalises the Gaussian Malliavin calculus to Wigner densities, and allows one to prove the smoothness of joint Wigner distributions with Gaussian marginals using Sobolev spaces over $\mathbb{R}^2$. Here, noncommutative Gaussian processes can be built as the couple of the position and momentum Brownian motions on the Fock space. We also provide a treatment of other probability laws, including noncommutative couples of random variables with gamma and continuous binomial marginals based on the affine algebra. More generally, the long term goal in this field is to extend the hypoellipticity results of the Malliavin calculus to noncommutative quantum processes. In this chapter, we also point out the relationship between noncommutative and commutative differential calculi. In the white noise case, *i.e.*, if the underlying Hilbert space is the L^2-space of some measure space, the classical divergence operator defines an anticipating stochastic integral, known as the Hitsuda–Skorohod integral.

Several books on other extensions of the Malliavin calculus have been recently published, such as [86] which deals with infinitesimal (nonstandard) analysis and [56] which deals with Lévy processes. See [108] for a recent introduction to quantum stochastic calculus with connections to noncommutative geometry. See also [26] and [123] for recent introductions to quantum stochastics based on quantum stochastic calculus and quantum Markov semigroups.

The outline of the book is as follows (we refer the reader to [55] for an introduction to the basic concepts of quantum theory used in this book). In Chapter 1 we introduce the boson Fock space and we show how the first moments of the associated normal distribution can be computed using basic noncommutative calculus. Chapter 2 collects the background material on real Lie algebras and their representations. In Chapter 3 we consider fundamental examples of probability distributions (Gaussian, Poisson, gamma), and their connections with the Heisenberg–Weyl and oscillator algebras, as well as with the special linear algebra $\mathfrak{sl}_2(\mathbb{R})$, generated by the annihilation and creation operators on the boson Fock space. This will also be the occasion to introduce other representations based on polynomials. After those introductory sections, the construction of noncommutative random variables as operators acting on Lie algebras will be formalised in Chapter 4, based in particular on the notion of spectral measure. Quantum stochastic integrals are introduced in Chapter 5. In Chapter 6 we revisit the approaches of Chapters 3 and 4, relating Lie algebraic relations and probability distributions, in the unified framework of the splitting lemma, see chapter 1 of [38]. The problem of defining joint densities of couples of noncommutative random variables is treated in Chapter 7 under the angle of Weyl calculus, and Lévy processes on real Lie algebras are considered in Chapter 8. The classical, commutative Malliavin calculus is introduced in Chapter 9, and an introduction to quasi-invariance and the Girsanov theorem for noncommutative Lévy processes is given in Chapter 10. The noncommutative counterparts of the Malliavin calculus for Gaussian distributions, and then for gamma and other related probability densities are treated in Chapters 11 and 12, respectively, including the case of $so(3)$.