CAMBRIDGE
Cambridge University Press
978-1-107-12844-6 — Erdős–Ko–Rado Theorems: Algebraic Approaches
Christopher Godsil , Karen Meagher
Excerpt
More Information

1

The Erdős–Ko–Rado Theorem

A family of subsets of a set is *intersecting* if any two elements of the family have at least one element in common. It is easy to find small intersecting families; the basic problem is to decide how large they can be, and to describe the structure of the families that meet whatever bound we can derive. The prototype of all these results is the following, which is often referred to as the *Erdős–Ko–Rado Theorem for intersecting families*.

1.0.1 Theorem. *Let k and n be integers with $n \geq 2k$. If $\mathcal{A}$ is a collection of k-subsets of the set $\{1, \dots, n\}$, such that any two members of $\mathcal{A}$ have at least one point in common, then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

Moreover, if $n > 2k$, equality holds if and only if $\mathcal{A}$ consists of all the k-subsets that contain a given point from $\{1, \dots, n\}$. □

This result provides both a bound and a characterization; the bound is easier to prove, but the characterization can be more useful. This theorem is essentially a corollary of a stronger result first proved by Erdős, Ko and Rado [58]. Define a *t-intersecting family* to be a collection of subsets such that any two have at least t points in common. Then the *Erdős–Ko–Rado Theorem for t-intersecting families* can be stated as follows:

1.0.2 Theorem. *Let n, k and t be positive integers with $0 \leq t \leq k$. There exists a function $f(k, t)$ such that if $n \geq f(k, t)$ and $\mathcal{A}$ is a t-intersecting family of k-sets of $\{1, \dots, n\}$, then*

$$|\mathcal{A}| \leq \binom{n-t}{k-t}.$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-107-12844-6 — Erdős–Ko–Rado Theorems: Algebraic Approaches
Christopher Godsil , Karen Meagher
Excerpt
More Information

Moreover, equality holds if and only if $\mathcal{A}$ consists of all k-subsets that contain a specified t-subset of $\{1, \dots, n\}$.

This theorem is clearly more general than our first theorem, but has the weakness that its conclusion holds only if n is greater than some unspecified lower bound.

We call a collection of subsets of $\{1, \dots, n\}$ a *set system* with underlying set $\{1, \dots, n\}$; if the subsets all have size k we refer to it as a *k-set system*. The easiest way to build a t-intersecting k-set system on an n-set is to simply take all k-subsets that contain a fixed t-set; clearly such a system has size

$$\binom{n-t}{k-t}.$$

We call a set system of this type a *canonical t-intersecting family*. The lower bound on n in Theorem 1.0.2 is necessary because when n is not large enough, an intersecting family of maximal size need not be canonical. Much work was devoted to determining the precise value of $f(n, k)$ needed. Examples show that we need $n \geq (t+1)(k-t+1)$ for the bound to hold, and in 1978 Frankl [62] proved that this constraint sufficed when t was large enough. In 1984, Wilson [172] proved that the bound in the EKR Theorem holds if $n \geq (t+1)(k-t+1)$, and the characterization holds provided $n > (t+1)(k-t+1)$. (We present his proof in Chapter 6.) In 1997 Ahlswede and Khachatrian [3] determined the largest t-intersecting k-set systems on an n-set, for all values of n. The result of this work is that, for each choice of n, k and t, we know that maximum size of the t-intersecting families and we know the structure of the families that reach this size.

In this chapter we present the original proof of the EKR Theorem, discuss the work of Ahlswede and Khachatrian, and then provide a number of different proofs of the EKR Theorem for intersecting families.

1.1 The original proof

In their paper, Erdős, Ko and Rado prove two statements. The first was the bound in Theorem 1.0.1, and the second was Theorem 1.0.2, with a very rough upper bound on the value of the function $f(k, t)$. In this section we present the original proofs of both statements. Both proofs use a simple yet useful operation on set systems called *shifting* or *set compression*. For a comprehensive overview of this method see the survey paper by Frankl [68].

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-12844-6 — Erdős–Ko–Rado Theorems: Algebraic Approaches
Christopher Godsil , Karen Meagher
Excerpt
More Information

Let $\mathcal{A}$ be a k-set system on an n-set. For any integers i, j from $\{1, \ldots, n\}$, define the (i, j)*-shift of a set* A in $\mathcal{A}$ by

$$S_{i,j}(A) = \begin{cases} (A \setminus \{j\}) \cup \{i\}, & \text{if } j \in A \text{ and } i \notin A \text{ and } (A \setminus \{j\}) \cup \{i\} \notin \mathcal{A}; \\ A, & \text{otherwise.} \end{cases}$$

Then the (i, j)*-shift of a set system* $\mathcal{A}$ is defined by

$$S_{i,j}(\mathcal{A}) = \{S_{i,j}(A) : A \in \mathcal{A}\}.$$

Obviously if $\mathcal{A}$ is a k-set system then $S_{i,j}(\mathcal{A})$ is also a k-set system of the same size. What is more interesting is that shifting a t-intersecting set system results in a system that is still t-intersecting.

1.1.1 Lemma. *If $\mathcal{A}$ is a t-intersecting system of k-sets, then for any distinct pair i, j the set system $S_{i,j}(\mathcal{A})$ is also t-intersecting.*

Proof. Let A and B be any two sets in $\mathcal{A}$. We will show that

$$|S_{i,j}(A) \cap S_{i,j}(B)| \geq t.$$

If both sets remain unchanged by the shifting operation or if both sets are changed by the shifting operation, then clearly $S_{i,j}(A)$ and $S_{i,j}(B)$ are still t-intersecting.

Thus we can assume that $S_{i,j}(A) = A$ and $S_{i,j}(B) \neq B$. Since the shifting operation changes the set B, we know that $S_{i,j}(B) = (B \setminus \{j\}) \cup \{i\}$. As the set A does not change, then at least one of the following holds:

(a) $j \notin A$,
(b) $i \in A$,
(c) $(A \setminus \{j\}) \cup \{i\} \in \mathcal{A}$.

Since A and B are t-intersecting, if either (a) or (b) holds, then it is easy to see that $S_{i,j}(A)$ (which equals A) and $S_{i,j}(B)$ are t-intersecting. If case (c) holds, then $(A \setminus \{j\}) \cup \{i\}$ is in $\mathcal{A}$, and it must be at least t-intersecting with B. Since j is not in $(A \setminus \{j\}) \cup \{i\}$ and i is not in B, neither i nor j are one of the t elements in the intersection of $(A \setminus \{j\}) \cup \{i\}$ and B. Thus, all of these t elements must also be in both $S_{i,j}(A) = A$ and $S_{i,j}(B)$. □

If for all $i < j$ the set system $\mathcal{A}$ has the property that $S_{i,j}(\mathcal{A}) = \mathcal{A}$, then $\mathcal{A}$ is called a *left-shifted system* (or a *stable system*). Any set system $\mathcal{A}$ can be transformed into a left-shifted set system by shifting; in fact this can be done in as few as $\binom{n}{2}$ shifts [68, Proposition 2.2].

We now have all the tools to prove the bound in the Erdős–Ko–Rado Theorem for $t = 1$.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-12844-6 — Erdős–Ko–Rado Theorems: Algebraic Approaches
Christopher Godsil , Karen Meagher
Excerpt
More Information

1.1.2 Theorem. *Let k and n be integers with $n \geq 2k$. If $\mathcal{A}$ is an intersecting k-set system on an n-set, then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

Proof. First consider the case where $2k = n$. Since $\mathcal{A}$ is intersecting, it is not possible that both a set and its complement are in $\mathcal{A}$, and thus

$$\mathcal{A} \leq \frac{1}{2}\binom{n}{k} = \binom{n-1}{k-1}.$$

Next consider the case where $2k < n$ and apply induction on n. If $n = 3$, then $k = 1$, so clearly $|\mathcal{A}| = 1$ and the theorem holds. Assume the theorem holds for $n_0 < n$. Let $\mathcal{A}$ be an intersecting k-set system on an n-set and further assume that $\mathcal{A}$ is a left-shifted system. In particular, $S_{i,n}(\mathcal{A}) = \mathcal{A}$ for all $i < n$.

The system $\mathcal{A}$ can be split into two subsystems: the first is the collection of all sets that do not contain n, and the second is all sets that do contain n. The first subsystem is an intersecting k-set system on an $(n-1)$-set and by induction has size no more than $\binom{n-2}{k-1}$. We claim that the size of the second subsystem is no more than $\binom{n-2}{k-2}$, which implies that

$$|\mathcal{A}| \leq \binom{n-2}{k-2} + \binom{n-2}{k-1} = \binom{n-1}{k-1}$$

and completes the proof of the theorem.

To prove this claim, we show that if n is removed from each set in the second subsystem, what remains is an intersecting $(k-1)$-set system on an $(n-1)$-set, which by induction implies that the size of the second subsystem is no more than $\binom{n-2}{k-2}$.

Thus, we only need to show for any two sets A_1 and A_2 from the second subsystem that

$$(A_1 \setminus \{n\}) \cap (A_2 \setminus \{n\}) \neq \emptyset.$$

Since $n > 2k$, there is an $x \in \{1, \dots, n\}$ that is not in $A_1 \cup A_2$. Further, the set $\mathcal{A}$ is left-shifted, and hence the set

$$B_1 = (A_1 \setminus \{n\}) \cup \{x\}$$

is in $\mathcal{A}$. The set system $\mathcal{A}$ is intersecting, so this means that $|A_2 \cap B_1| \geq 1$ and that the sets A_1 and A_2 have a point in common that is not n. □

The bound $n \geq 2k$ is both necessary and the best possible, since if $n < 2k$ then any two k-sets from an n-set are intersecting.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-107-12844-6 — Erdős–Ko–Rado Theorems: Algebraic Approaches
Christopher Godsil , Karen Meagher
Excerpt
More Information

Next we sketch the original proof given by Erdős, Ko and Rado that a t-intersecting k-set system that is not a canonical t-intersecting k-set system is strictly smaller than $\binom{n-t}{k-t}$, provided that n is sufficiently large relative to k and t.

1.1.3 Theorem. *Let $\mathcal{A}$ be a t-intersecting k-set system on an n-set. If $n \geq t + (k-t)\binom{k}{t}^3$ and $\mathcal{A}$ is not a canonical t-intersecting set system, then*

$$|\mathcal{A}| < \binom{n-t}{k-t}.$$

Proof. Set $r = |\bigcap_{A\in\mathcal{A}} A|$; since $\mathcal{A}$ is not a canonical t-intersecting set system we see that $r < t$.

Provided that $n \geq 2k - t$, a maximal t-intersecting set system is not a $(t+1)$-intersecting system (this fact is left as an exercise), so there must be sets $A_1, A_2 \in \mathcal{A}$ such that $|A_1 \cap A_2| = t$. Since $\mathcal{A}$ is not a canonical t-intersecting system there also must be a set $A_3 \in \mathcal{A}$ such that

$$|A_1 \cap A_2 \cap A_3| < t.$$

Consider the collection T of triples (B_1, B_2, B_3) of t-subsets with $B_i \subset A_i$ for $i = 1, 2, 3$ such that

$$t < |B_1 \cup B_2 \cup B_3| \leq k.$$

For each triple (B_1, B_2, B_3) in T, define a system of sets by

$$\Phi_{(B_1,B_2,B_3)} = \{A : (B_1 \cup B_2 \cup B_3) \subseteq A \in \mathcal{A}\}.$$

Every set in $\mathcal{A}$ must have intersection of size at least t with each of A_1, A_2 and A_3; thus every set in $\mathcal{A}$ must contain at least one set $B_1 \cup B_2 \cup B_3$ where $(B_1, B_2, B_3) \in T$. Thus, the union of the sets $\Phi_{(B_1,B_2,B_3)}$ over all the triples from T is exactly $\mathcal{A}$. So by bounding the size of T and the size of any $\Phi_{(B_1,B_2,B_3)}$, we get a bound on the size of $\mathcal{A}$.

A very rough (but still effective) upper bound on the size of T is $\binom{k}{t}^3$. Further, if we assume that $s = |B_1 \cup B_2 \cup B_3|$, then an upper bound on the size of $\Phi_{(B_1,B_2,B_3)}$ is $\binom{n-s}{k-s}$. Since $s \geq t+1$, this bound is no larger than $\binom{n-t-1}{k-t-1}$. Putting these together, we have the bound

$$|\mathcal{A}| = \left| \bigcup_{(B_1,B_2,B_3)\in T} \Phi_{(B_1,B_2,B_3)} \right| \leq \binom{n-t-1}{k-t-1}\binom{k}{t}^3.$$

Finally, if $n > t + (k-t)\binom{k}{t}^3$ then this upper bound is less than $\binom{n-t}{k-t}$ and the theorem holds. □

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-12844-6 — Erdős–Ko–Rado Theorems: Algebraic Approaches
Christopher Godsil , Karen Meagher
Excerpt
More Information

In the previous proof the lower bound on the size of n is

$$n \geq t + (k-t)\binom{k}{t}^3.$$

Erdős, Ko and Rado state in their paper that this is clearly not the optimal lower bound on n, but that a bound on n is needed. They give the following example of a 2-intersecting 4-set system on an 8-set system that is strictly larger than the canonical intersecting set system:

$$\begin{array}{lllll}\{\{1,2,3,4\}, & \{1,2,3,5\}, & \{1,2,3,6\}, & \{1,2,3,7\}, & \{1,2,3,8\}, \\ \ \{1,2,4,5\}, & \{1,2,4,6\}, & \{1,2,4,7\}, & \{1,2,4,8\}, & \{1,3,4,5\}, \\ \ \{1,3,4,6\}, & \{1,3,4,7\}, & \{1,3,4,8\}, & \{2,3,4,5\}, & \{2,3,4,6\}, \\ \ \{2,3,4,7\}, & \{2,3,4,8\}\ \} & & & \end{array}$$

Figure 1.1 A 2-intersecting 4-set system on an 8-set

This system has size 17, whereas the canonical 2-intersecting set system has only $\binom{8-2}{4-2} = 15$ sets. There is a simple pattern to this system: it is the collection of every 4-set that has at least 3 elements from the set $\{1, 2, 3, 4\}$. Such systems are discussed in more detail in the next section.

1.2 Non-canonical intersecting set systems

In Figure 1.1 we give an example of a 2-intersecting set system that is larger than the canonical 2-intersecting set system. This example can be generalized to provide a large family of t-intersecting set systems that are strictly larger than the canonical t-intersecting set system. Consider the system of all $2r$-subsets from a $4r$-set that have at least $r+1$ elements from the set $\{1, \ldots, 2r\}$. It is not hard to see that this is a 2-intersecting set system of size

$$\sum_{i=r+1}^{2r} \binom{2r}{i}\binom{2r}{2r-i} = \frac{1}{2}\left(\binom{4r}{2r} - \binom{2r}{r}^2\right).$$

This is larger than the size of a canonical 2-intersecting system for all $r > 1$.

In fact if $n < (k-t+1)(t+1)$ and $t > 1$, there is a t-intersecting k-set system on an n-set that is strictly larger than the canonical system. It is easy to describe these systems. Assume that n, k and t are integers with $t \leq k \leq n$. For $i \in \{0, \ldots, k-t\}$ define

$$\mathcal{F}_i = \{A \ :\ |A| = k,\ |A \cap \{1, \ldots, 2i+t\}| \geq i+t\}.$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-107-12844-6 — Erdős–Ko–Rado Theorems: Algebraic Approaches
Christopher Godsil , Karen Meagher
Excerpt
More Information

If $i \in \{0, \dots, k-t\}$, the system $\mathcal{F}_i$ is a t-intersecting k-set system on an n-set and $\mathcal{F}_0$ is a canonical t-intersecting system. The size of $\mathcal{F}_i$ is

$$|\mathcal{F}_i| = \sum_{j=t+i}^{t+2i} \binom{t+2i}{j}\binom{n-(t+2i)}{k-j}. \tag{1.2.1}$$

Ahlswede and Khachatrian [3] determined bounds on n (relative to k and t) for when $\mathcal{F}_{i+1}$ is larger than $\mathcal{F}_i$. Further, they prove that for all n one of the systems $\mathcal{F}_i$ is the largest intersecting system. This impressive result is stated as Theorem 1.3.1 in the next section; we give one case of this theorem in the following lemma.

1.2.1 Lemma. *If $n = (k-t+1)(t+1)$, then $|\mathcal{F}_0| = |\mathcal{F}_1|$ and if $n < (k-t+1)(t+1)$, then $|\mathcal{F}_0| < |\mathcal{F}_1|$.*

Proof. The system $\mathcal{F}_1$ is the collection of all k-subsets from an n-set that contain at least $t+1$ elements from the set $\{1, \dots, t+2\}$. From (1.2.1),

$$|\mathcal{F}_1| = \binom{t+2}{t+1}\binom{n-(t+2)}{k-(t+1)} + \binom{t+2}{t+2}\binom{n-(t+2)}{k-(t+2)}.$$

Since

$$|\mathcal{F}_0| = \binom{n-t}{k-t} = \binom{n-t-2}{k-t} + 2\binom{n-t-2}{k-t-1} + \binom{n-t-2}{k-t-2}$$

it follows that

$$\begin{aligned}\binom{n-t}{k-t} - |\mathcal{F}_1| &= \binom{n-t-2}{k-t} - t\binom{n-t-2}{k-t-1}\\ &= \binom{n-t-2}{k-t-1}\left(\frac{n-k-1}{k-t} - t\right)\\ &= \binom{n-t-2}{k-t-1}\left(\frac{n-(t+1)(k-t+1)}{k-t}\right).\end{aligned}$$ □

With this simple example we can see that the lower bound of $f(k,t) = (k-t+1)(t+1)$ on n in Theorem 1.0.2 cannot be improved. It is considerably more difficult to prove for $n > (k-t+1)(t+1)$ that the only systems of maximum size are the canonical intersecting set systems.

1.3 The complete Erdős–Ko–Rado Theorem

Theorem 1.0.2 describes the size and the structure of the largest intersecting set systems, provided that the size of the underlying set is large relative to the size of the subsets and the size of the intersection. Ahlswede and Khachatrian [3]

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-12844-6 — Erdős–Ko–Rado Theorems: Algebraic Approaches
Christopher Godsil , Karen Meagher
Excerpt
More Information

proved in 1997 what they called the "Complete Erdős–Ko–Rado Theorem." This theorem gives the size and structure of the maximum t-intersecting k-set systems for all values of n.

The characterization is up to isomorphism: two set systems on an n-set are *isomorphic* if one system can be can be obtained from the other by a permutation of the underlying n-set. Any two of the canonical t-intersecting k-set systems are isomorphic (and these are the only systems isomorphic to the canonical t-intersecting systems).

1.3.1 Theorem. *Let t, k, n be positive integers with $1 \le t \le k \le n$ and r be an integer with $r \le k - t$. If*

$$(k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right), \tag{1.3.1}$$

then $\mathcal{F}_r$ is the unique (up to isomorphism) t-intersecting k-set system with maximal cardinality. (By convention, $\frac{t-1}{r} = \infty$ for $r = 0$.) If

$$n = (k-t+1)\left(2+\frac{t-1}{r+1}\right), \tag{1.3.2}$$

then $|\mathcal{F}_r| = |\mathcal{F}_{r+1}|$, and these systems are the unique (up to isomorphism) t-intersecting k-set systems with maximal cardinality.

This theorem covers all relevant values of n, since if $r = k - t + 1$ then $n < 2k - t + 1$ and any two k-sets would be t-intersecting. Further, this theorem includes Theorem 1.0.2 with the exact lower bound on n. In particular, if $n > (k - t + 1)(t + 1)$ then $r = 0$ in inequality (1.3.1), and the canonical t-intersecting system $\mathcal{F}_0$ is the largest system.

The proof of the Erdős–Ko–Rado Theorem for t-intersecting families is much more complicated than the proof of the Erdős–Ko–Rado Theorem for intersecting families. The only known proof of the general result, given by Ahlswede and Khachatrian [3], also uses the shifting operation defined in Section 1.1. We do not include the proof here, but we refer the reader to the book *Lectures on Advances in Combinatorics* by Ahlswede and Blinovsky [2], where the proof of this theorem is presented in Lecture 1. We give two examples to help illustrate what this theorem means.

The example from Figure 1.1 is a 2-intersecting 4-set system on an 8-set (so $t = 2$, $k = 4$ and $n = 8$). The value of r that satisfies

$$(k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right)$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-107-12844-6 — Erdős–Ko–Rado Theorems: Algebraic Approaches
Christopher Godsil , Karen Meagher
Excerpt
More Information

for these values of n, k and t is $r = 1$. This means that the largest 2-intersecting 4-set system on an 8-set is isomorphic to the system of 4-sets

$$\mathcal{F}_1 = \{A : |A \cap \{1, 2, 3, 4\}| \geq 3\},$$

which has size

$$\binom{4}{3}\binom{4}{1} + \binom{4}{4} = 17.$$

If $t = 1$, the inequality (1.3.1) in Theorem 1.3.1 reduces to

$$2k < n < 2\left(k + \frac{0}{r}\right).$$

If $n > 2k$ this is only satisfied when $r = 0$ (with the convention that $\frac{0}{0} = \infty$) and the unique maximum system is $\mathcal{F}_0$, the canonical intersecting system.

On the other hand, if $n = 2k$ then (1.3.2) in the complete Erdős–Ko–Rado theorem reduces to

$$n = k\left(2 + \frac{0}{r+1}\right)$$

which is satisfied for any value of $r \in \{0, \ldots, k-1\}$. In this case there are many non-isomorphic systems that meet the bound of $\binom{n-1}{k-1}$, since the system $\mathcal{F}_r$ is a maximum system for all $r \in \{0, \ldots, k-1\}$. To see that all these systems have the same size, consider

$$\begin{aligned} |\mathcal{F}_r| &= \sum_{i=r+1}^{2r+1} \binom{2r+1}{i}\binom{2k-(2r+1)}{k-i} \\ &= \frac{1}{2}\sum_{i=0}^{2r+1} \binom{2r+1}{i}\binom{2k-2r-1}{k-i} \qquad (1.3.3) \\ &= \frac{1}{2}\binom{2k}{k} \\ &= \binom{2k-1}{k-1}. \end{aligned}$$

We see that (1.3.3) holds since

$$\begin{aligned} \sum_{i=r+1}^{2r+1} \binom{2r+1}{i}\binom{2k-2r-1}{k-i} &= \sum_{i=r+1}^{2r+1} \binom{2r+1}{2r+1-i}\binom{2k-2r-1}{k+i-2r-1} \\ &= \sum_{i=0}^{r} \binom{2r+1}{i}\binom{2k-2r-1}{k-i}. \end{aligned}$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-12844-6 — Erdős–Ko–Rado Theorems: Algebraic Approaches
Christopher Godsil , Karen Meagher
Excerpt
More Information

In particular, the collection of all k-subsets that contain 1 form an intersecting set system of this size; this is isomorphic to the set system $\mathcal{F}_0$. Also, the collection of all k-subsets that contain at least two elements from the set $\{1, 2, 3\}$ is of this size. Further, the system of all sets that do not contain 1 also form an intersecting set system that is isomorphic to the system $\mathcal{F}_{k-1}$.

1.4 The shifting technique

Shortly after the original proof of the Erdős–Ko–Rado theorem appeared, there was a rush to find shorter and more elegant proofs of the result, particularly for the case when $t = 1$. In this, and the following section, two of these proofs are outlined (and a third is outlined in the Exercises). In Section 2.14, Section 6.6 and Section 7.1, three more more proofs of this result are given.

Our first alternate proof of the Erdős–Ko–Rado theorem appears in Frankl's paper on the shifting technique in extremal set theory [68]. This proof only establishes the bound in the theorem, but this method can be generalized to a nice proof of the Hilton–Milner Theorem (see Section 1.6.)

1.4.1 Lemma. *Suppose $\mathcal{A}$ is a left-shifted intersecting k-set system. Then for all $A, B \in \mathcal{A}$,*

$$A \cap B \cap \{1, \dots, 2k-1\} \neq \emptyset.$$

Proof. If this statement does not hold, consider sets A and B with

$$A \cap B \cap \{1, \dots, 2k-1\} = \emptyset,$$

where $|A \cap \{1, \dots, 2k-1\}|$ is maximized. Then there exists a $j \in A \cap B$ with $j > 2k-1$. Further, since $|A \setminus \{j\}| = k-1$ and $|B \setminus \{j\}| = k-1$, there is an $i \leq 2k-1$ with $i \notin A \cup B$. Since the system $\mathcal{A}$ is left-shifted, the set $(A \setminus \{j\}) \cup \{i\}$ is in $\mathcal{A}$ and has

$$((A \setminus \{j\}) \cup \{i\}) \cap B \cap \{1, \dots, 2k-1\} = \emptyset,$$

but this contradicts the maximality of $|A \cap \{1, \dots, 2k-1\}|$ among all such sets. $\square$

This proposition can be used to prove the bound in the Erdős–Ko–Rado theorem when $t = 1$ and $n \geq 2k$. We apply induction on the size of k. Clearly, if $k = 1$ then the statement is trivial, and if $n = 2k$ it follows easily since $\mathcal{A}$ cannot contain both a set and its complement.

For $n > 2k$, assume that $\mathcal{A}$ is a left-shifted intersecting k-set system. Define

$$\mathcal{A}_i = \{A \cap \{1, \dots, 2k\} : A \in \mathcal{A},\ |A \cap \{1, \dots, 2k\}| = i\}.$$

© in this web service Cambridge University Press www.cambridge.org