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Recurrence

Ergodic theory studies the behavior of dynamical systems with respect to measures that remain invariant under time evolution. Indeed, it aims to describe those properties that are valid for the trajectories of almost all initial states of the system, that is, all but a subset that has zero weight for the invariant measure. Our first task, in Section 1.1, will be to explain what we mean by 'dynamical system' and 'invariant measure'.

The roots of the theory date back to the first half of the 19th century. By 1838, the French mathematician Joseph Liouville observed that every energy-preserving system in classical (Newtonian) mechanics admits a natural invariant volume measure in the space of configurations. Just a bit later, in 1845, the great German mathematician Carl Friedrich Gauss pointed out that the transformation

$$(0,1] \to \mathbb{R}, \quad x \mapsto \text{fractional part of } \frac{1}{x},$$

which has an important role in number theory, admits an invariant measure equivalent to the Lebesgue measure (in the sense that the two have the same zero measure sets). These are two of the examples of applications of ergodic theory that we discuss in Section 1.3. Many others are introduced throughout this book.

The first important result was found by the great French mathematician Henri Poincaré by the end of the 19th century. Poincaré was particularly interested in the motion of celestial bodies, such as planets and comets, which is described by certain differential equations originating from Newton's law of universal gravitation. Starting from Liouville's observation, Poincaré realized that for almost every initial state of the system, that is, almost every value of the initial position and velocity, the solution of the differential equation comes back arbitrarily close to that initial state, unless it goes to infinity. Even more, this *recurrence* property is not specific to (celestial) mechanics: it is shared by any dynamical system that admits a finite invariant measure. That is the theme of Section 1.2.

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The same theme reappears in Section 1.5, in a more elaborate context: there, we deal with any finite number of dynamical systems commuting with each other, and we seek *simultaneous* returns of the orbits of all those systems to the neighborhood of the initial state. This kind of result has important applications in combinatorics and number theory, as we will see.

The recurrence phenomenon is also behind the constructions that we present in Section 1.4. The basic idea is to fix some positive measure subset of the domain and to consider the first return to that subset. This first-return transformation is often easier to analyze, and it may be used to shed much light on the behavior of the original transformation.

1.1 Invariant measures

Let $(M,\mathcal{B},\mu)$ be a measure space and $f : M \to M$ be a measurable transformation. We say that the measure μ is *invariant* under f if

$$\mu(E) = \mu(f^{-1}(E)) \quad \text{for every measurable set } E \subset M. \tag{1.1.1}$$

We also say that μ is f*-invariant*, or that f *preserves* μ, to mean just the same. Notice that the definition (1.1.1) makes sense, since the pre-image of a measurable set under a measurable transformation is still a measurable set. Heuristically, the definition means that the probability that a point picked "at random" is in a given subset is equal to the probability that its image is in that subset.

It is possible, and convenient, to extend this definition to other types of dynamical systems, beyond transformations. We are especially interested in *flows*, that is, families of transformations $f^t : M \to M$, with $t \in \mathbb{R}$, satisfying the following conditions:

$$f^0 = \mathrm{id} \quad \text{and} \quad f^{s+t} = f^s \circ f^t \text{ for every } s,t \in \mathbb{R}. \tag{1.1.2}$$

In particular, each transformation f^t is invertible and the inverse is f^{-t}. Flows arise naturally in connection with differential equations of the form

$$\frac{d\gamma}{dt}(t) = X(\gamma(t))$$

in the following way: under suitable conditions on the vector field X, for each point x in the domain M there exists exactly one solution $t \mapsto \gamma_x(t)$ of the differential equation with $\gamma_x(0) = x$; then $f^t(x) = \gamma_x(t)$ defines a flow in M.

We say that a measure μ is *invariant* under a flow $(f^t)_t$ if it is invariant under each one of the transformations f^t, that is, if

$$\mu(E) = \mu(f^{-t}(E)) \quad \text{for every measurable set } E \subset M \text{ and } t \in \mathbb{R}. \tag{1.1.3}$$

Proposition 1.1.1. *Let $f : M \to M$ be a measurable transformation and μ be a measure on M. Then f preserves μ if and only if*

$$\int \phi \, d\mu = \int \phi \circ f \, d\mu \tag{1.1.4}$$

for every μ-integrable function $\phi : M \to \mathbb{R}$.

Proof. Suppose that the measure μ is invariant under f. We are going to show that the relation (1.1.4) is valid for increasingly broader classes of functions. Let $\mathcal{X}_B$ denote the characteristic function of any measurable subset B. Then

$$\mu(B) = \int \mathcal{X}_B \, d\mu \quad \text{and} \quad \mu(f^{-1}(B)) = \int \mathcal{X}_{f^{-1}(B)} \, d\mu = \int (\mathcal{X}_B \circ f) \, d\mu.$$

Thus, the hypothesis $\mu(B) = \mu(f^{-1}(B))$ means that (1.1.4) is valid for characteristic functions. Then, by linearity of the integral, (1.1.4) is valid for all simple functions. Next, given any integrable $\phi : M \to \mathbb{R}$, consider a sequence $(s_n)_n$ of simple functions, converging to ϕ and such that $|s_n| \le |\phi|$ for every n. That such a sequence exists is guaranteed by Proposition A.1.33. Then, using the dominated convergence theorem (Theorem A.2.11) twice:

$$\int \phi \, d\mu = \lim_n \int s_n \, d\mu = \lim_n \int (s_n \circ f) \, d\mu = \int (\phi \circ f) \, d\mu.$$

This shows that (1.1.4) holds for every integrable function if μ is invariant. The converse is also contained in the arguments we just presented.

1.1.1 Exercises

1.1.1. Let $f : M \to M$ be a measurable transformation. Show that a Dirac measure δ_p is invariant under f if and only if p is a fixed point of f. More generally, a probability measure $\delta_{p,k} = k^{-1}\big(\delta_p + \delta_{f(p)} + \cdots + \delta_{f^{k-1}(p)}\big)$ is invariant under f if and only if $f^k(p) = p$.

1.1.2. Prove the following version of Proposition 1.1.1. Let M be a metric space, $f : M \to M$ be a measurable transformation and μ be a measure on M. Show that f preserves μ if and only if

$$\int \phi \, d\mu = \int \phi \circ f \, d\mu$$

for every bounded continuous function $\phi : M \to \mathbb{R}$.

1.1.3. Prove that if $f : M \to M$ preserves a measure μ then, given any $k \ge 2$, the iterate f^k also preserves μ. Is the converse true?

1.1.4. Suppose that $f : M \to M$ preserves a probability measure μ. Let $B \subset M$ be a measurable set satisfying any one of the following conditions:

(a) $\mu(B \setminus f^{-1}(B)) = 0$;
(b) $\mu(f^{-1}(B) \setminus B) = 0$;
(c) $\mu(B \Delta f^{-1}(B)) = 0$;
(d) $f(B) \subset B$.

Show that there exists $C \subset M$ such that $f^{-1}(C) = C$ and $\mu(B \Delta C) = 0$.

1.1.5. Let $f : U \to U$ be a C^1 diffeomorphism on an open set $U \subset \mathbb{R}^d$. Show that the Lebesgue measure m is invariant under f if and only if $|\det Df| \equiv 1$.

1.2 Poincaré recurrence theorem

We are going to study two versions of Poincaré's theorem. The first one (Section 1.2.1) is formulated in the context of (finite) measure spaces. The theorem of Kač, that we state and prove in Section 1.2.2, provides a quantitative complement to that statement. The second version of the recurrence theorem (Section 1.2.3) assumes that the ambient is a topological space with certain additional properties. We will also prove a third version of the recurrence theorem, due to Birkhoff, whose statement is purely topological.

1.2.1 Measurable version

Our first result asserts that, given any *finite* invariant measure, almost every point in any positive measure set E returns to E an infinite number of times:

Theorem 1.2.1 (Poincaré recurrence)**.** *Let $f : M \to M$ be a measurable transformation and μ be a finite measure invariant under f. Let $E \subset M$ be any measurable set with $\mu(E) > 0$. Then, for μ-almost every point $x \in E$ there exist infinitely many values of n for which $f^n(x)$ is also in E.*

Proof. Denote by E_0 the set of points $x \in E$ that never return to E. As a first step, let us prove that E_0 has zero measure. To this end, let us observe that the pre-images $f^{-n}(E_0)$ are pairwise disjoint. Indeed, suppose there exist $m > n \geq 1$ such that $f^{-m}(E_0)$ intersects $f^{-n}(E_0)$. Let x be a point in the intersection and $y = f^n(x)$. Then $y \in E_0$ and $f^{m-n}(y) = f^m(x) \in E_0$. Since $E_0 \subset E$, this means that y returns to E at least once, which contradicts the definition of E_0. This contradiction proves that the pre-images are pairwise disjoint, as claimed.

Since μ is invariant, we also have that $\mu(f^{-n}(E_0)) = \mu(E_0)$ for all $n \geq 1$. It follows that

$$\mu\left(\bigcup_{n=1}^{\infty} f^{-n}(E_0)\right) = \sum_{n=1}^{\infty} \mu(f^{-n}(E_0)) = \sum_{n=1}^{\infty} \mu(E_0).$$

The expression on the left-hand side is finite, since the measure μ is assumed to be finite. On the right-hand side we have a sum of infinitely many terms that are all equal. The only way such a sum can be finite is if the terms vanish. So, $\mu(E_0) = 0$ as claimed.

Now let us denote by F the set of points $x \in E$ that return to E a finite number of times. It is clear from the definition that every point $x \in F$ has some iterate

$f^k(x)$ in E_0. In other words,

$$F \subset \bigcup_{k=0}^{\infty} f^{-k}(E_0).$$

Since $\mu(E_0) = 0$ and μ is invariant, it follows that

$$\mu(F) \le \mu\Big(\bigcup_{k=0}^{\infty} f^{-k}(E_0)\Big) \le \sum_{k=0}^{\infty} \mu\big(f^{-k}(E_0)\big) = \sum_{k=0}^{\infty} \mu(E_0) = 0.$$

Thus, $\mu(F) = 0$ as we wanted to prove.

Theorem 1.2.1 implies an analogous result for continuous time systems: *if μ is a finite invariant measure of a flow $(f^t)_t$ then for every measurable set $E \subset M$ with positive measure and for μ-almost every $x \in E$ there exist times $t_j \to +\infty$ such that $f^{t_j}(x) \in E$.* Indeed, if μ is invariant under the flow then, in particular, it is invariant under the so-called *time*-1 *map* f^1. So, the statement we just made follows immediately from Theorem 1.2.1 applied to f^1 (the times t_j one finds in this way are integers). Similar observations apply to the other versions of the recurrence theorem that we present in the sequel.

On the other hand, the theorem in the next section is specific to discrete time systems.

1.2.2 Kač theorem

Let $f : M \to M$ be a measurable transformation and μ be a finite measure invariant under f. Let $E \subset M$ be any measurable set with $\mu(E) > 0$. Consider the *first-return time* function $\rho_E : E \to \mathbb{N} \cup \{\infty\}$, defined by

$$\rho_E(x) = \min\{n \ge 1 : f^n(x) \in E\} \tag{1.2.1}$$

if the set on the right-hand side is non-empty and $\rho_E(x) = \infty$ if, on the contrary, x has no iterate in E. According to Theorem 1.2.1, the second alternative occurs only on a set with zero measure.

The next result shows that this function is integrable and even provides the value of the integral. For the statement we need the following notation:

$$E_0 = \{x \in E : f^n(x) \notin E \text{ for every } n \ge 1\} \quad \text{and}$$
$$E_0^* = \{x \in M : f^n(x) \notin E \text{ for every } n \ge 0\}.$$

In other words, E_0 is the set of points in E that never *return* to E and E_0^* is the set of points in M that never *enter* E. We have seen in Theorem 1.2.1 that $\mu(E_0) = 0$.

Theorem 1.2.2 (Kač)**.** *Let $f : M \to M$ be a measurable transformation, μ be a finite invariant measure and $E \subset M$ be a positive measure set. Then the function*

ρ_E is integrable and

$$\int_E \rho_E \, d\mu = \mu(M) - \mu(E_0^*).$$

Proof. For each $n \geq 1$, define

$$E_n = \{x \in E : f(x) \notin E, \dots, f^{n-1}(x) \notin E, \text{ but } f^n(x) \in E\} \quad \text{and}$$
$$E_n^* = \{x \in M : x \notin E, f(x) \notin E, \dots, f^{n-1}(x) \notin E, \text{ but } f^n(x) \in E\}.$$

That is, E_n is the set of points of E that return to E for the first time exactly at time n,

$$E_n = \{x \in E : \rho_E(x) = n\},$$

and E_n^* is the set points that are not in E and enter E for the first time exactly at time n. It is clear that these sets are measurable and, hence, ρ_E is a measurable function. Moreover, the sets E_n, E_n^*, $n \geq 0$ constitute a *partition* of the ambient space: they are pairwise disjoint and their union is the whole of M. So,

$$\mu(M) = \sum_{n=0}^{\infty} \big(\mu(E_n) + \mu(E_n^*)\big) = \mu(E_0^*) + \sum_{n=1}^{\infty} \big(\mu(E_n) + \mu(E_n^*)\big). \tag{1.2.2}$$

Now observe that

$$f^{-1}(E_n^*) = E_{n+1}^* \cup E_{n+1} \quad \text{for every } n. \tag{1.2.3}$$

Indeed, $f(y) \in E_n^*$ means that the first iterate of $f(y)$ that belongs to E is $f^n(f(y)) = f^{n+1}(y)$ and that occurs if and only if $y \in E_{n+1}^*$ or else $y \in E_{n+1}$. This proves the equality (1.2.3). So, given that μ is invariant,

$$\mu(E_n^*) = \mu(f^{-1}(E_n^*)) = \mu(E_{n+1}^*) + \mu(E_{n+1}) \quad \text{for every } n.$$

Applying this relation successively, we find that

$$\mu(E_n^*) = \mu(E_m^*) + \sum_{i=n+1}^{m} \mu(E_i) \quad \text{for every } m > n. \tag{1.2.4}$$

The relation (1.2.2) implies that $\mu(E_m^*) \to 0$ when $m \to \infty$. So, taking the limit as $m \to \infty$ in the equality (1.2.4), we find that

$$\mu(E_n^*) = \sum_{i=n+1}^{\infty} \mu(E_i). \tag{1.2.5}$$

To complete the proof, replace (1.2.5) in the equality (1.2.2). In this way we find that

$$\mu(M) - \mu(E_0^*) = \sum_{n=1}^{\infty} \left(\sum_{i=n}^{\infty} \mu(E_i) \right) = \sum_{n=1}^{\infty} n\mu(E_n) = \int_E \rho_E \, d\mu,$$

as we wanted to prove.

In some cases, for example when the system (f,μ) is *ergodic* (this property will be defined and studied later, starting from Chapter 4), the set E_0^* has zero measure. Then the conclusion of the Kač theorem means that

$$\frac{1}{\mu(E)}\int_E \rho_E \, d\mu = \frac{\mu(M)}{\mu(E)} \tag{1.2.6}$$

for every measurable set E with positive measure. The left-hand side of this expression is the *mean return time* to E. So, (1.2.6) asserts that *the mean return time is inversely proportional to the measure of E.*

Remark 1.2.3. By definition, $E_n^* = f^{-n}(E) \setminus \bigcup_{k=0}^{n-1} f^{-k}(E)$. So, the fact that the sum (1.2.2) is finite implies that the measure of E_n^* converges to zero when $n \to \infty$. This fact will be useful later.

1.2.3 Topological version

Now let us suppose that M is a topological space, endowed with its Borel σ-algebra $\mathcal{B}$. A point $x \in M$ is *recurrent* for a transformation $f : M \to M$ if there exists a sequence $n_j \to \infty$ of natural numbers such that $f^{n_j}(x) \to x$. Analogously, we say that $x \in M$ is recurrent for a flow $(f^t)_t$ if there exists a sequence $t_j \to +\infty$ of real numbers such that $f^{t_j}(x) \to x$ when $j \to \infty$.

In the next theorem we assume that the topological space M admits a countable basis of open sets, that is, there exists a countable family $\{U_k : k \in \mathbb{N}\}$ of open sets such that every open subset of M may be written as a union of elements U_k of this family. This condition holds in most interesting examples.

Theorem 1.2.4 (Poincaré recurrence)**.** *Suppose that M admits a countable basis of open sets. Let $f : M \to M$ be a measurable transformation and μ be a finite measure on M invariant under f. Then, μ-almost every $x \in M$ is recurrent for f.*

Proof. For each k, denote by $\tilde{U}_k$ the set of points $x \in U_k$ that never return to U_k. According to Theorem 1.2.1, every $\tilde{U}_k$ has zero measure. Consequently, the countable union

$$\tilde{U} = \bigcup_{k \in \mathbb{N}} \tilde{U}_k$$

also has zero measure. Hence, to prove the theorem it suffices to check that every point x that is not in $\tilde{U}$ is recurrent. That is easy, as we are going to see. Consider $x \in M \setminus \tilde{U}$ and let U be any neighborhood of x. By definition, there exists some element U_k of the basis of open sets such that $x \in U_k$ and $U_k \subset U$. Since x is not in $\tilde{U}$, we also have that $x \notin \tilde{U}_k$. In other words, there exists $n \geq 1$ such that $f^n(x)$ is in U_k. In particular, $f^n(x)$ is also in U. Since the neighborhood U is arbitrary, this proves that x is a recurrent point.

Let us point out that the conclusions of Theorems 1.2.1 and 1.2.4 are false, in general, if the measure μ is not finite:

Example 1.2.5. Let $f : \mathbb{R} \to \mathbb{R}$ be the translation by 1, that is, the transformation defined by $f(x) = x + 1$ for every $x \in \mathbb{R}$. It is easy to check that f preserves the Lebesgue measure on $\mathbb{R}$ (which is infinite). On the other hand, no point $x \in \mathbb{R}$ is recurrent for f. According to the recurrence theorem, this last observation implies that f can not admit any finite invariant measure.

However, it is possible to extend these statements for certain cases of infinite measures: see Exercise 1.2.2.

To conclude, we present a purely topological version of Theorem 1.2.4, called the Birkhoff recurrence theorem, that makes no reference at all to invariant measures:

Theorem 1.2.6 (Birkhoff recurrence)**.** *If $f : M \to M$ is a continuous transformation on a compact metric space M then there exists some point $x \in X$ that is recurrent for f.*

Proof. Consider the family $\mathcal{I}$ of all non-empty closed sets $X \subset M$ that are invariant under f, in the sense that $f(X) \subset X$. This family is non-empty, since $M \in \mathcal{I}$. We claim that an element $X \in \mathcal{I}$ is minimal for the inclusion relation if and only if the orbit of every $x \in X$ is dense in X. Indeed, it is clear that if X is a closed invariant subset then X contains the closure of the orbit of each one of its elements. Hence, in order to be minimal, X must coincide with every one of these closures. Conversely, for the same reason, if X coincides with the orbit closure of each one of its points then it has no proper subset that is closed and invariant. That is, X is minimal. This proves our claim. In particular, every point x in a minimal set is recurrent. Therefore, to prove the theorem it suffices to prove that there exists some minimal set.

We claim that every totally ordered set $\{X_\alpha\} \subset \mathcal{I}$ admits a lower bound. Indeed, consider $X = \bigcap_\alpha X_\alpha$. Observe that X is non-empty, since the X_α are compact and they form a totally ordered family. It is clear that X is closed and invariant under f and it is equally clear that X is a lower bound for the set $\{X_\alpha\}$. That proves our claim. Now it follows from Zorn's lemma that $\mathcal{I}$ does contain minimal elements.

Theorem 1.2.6 can also be deduced from Theorem 1.2.4 together with the fact, which we will prove later (in Chapter 2), that every continuous transformation on a compact metric space admits some invariant probability measure.

1.2.4 Exercises

1.2.1. Show that the following statement is equivalent to Theorem 1.2.1, meaning that each one of them can be obtained from the other. Let $f : M \to M$ be a measurable transformation and μ be a finite invariant measure. Let $E \subset M$ be any measurable

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set with $\mu(E) > 0$. Then there exists $N \geq 1$ and a positive measure set $D \subset E$ such that $f^N(x) \in E$ for every $x \in D$.

1.2.2. Let $f : M \to M$ be an invertible transformation and suppose that μ is an invariant measure, not necessarily finite. Let $B \subset M$ be a set with finite measure. Prove that, given any measurable set $E \subset M$ with positive measure, μ-almost every point $x \in E$ either returns to E an infinite number of times or has only a finite number of iterates in B.

1.2.3. Let $f : M \to M$ be an invertible transformation and suppose that μ is a σ*-finite* invariant measure: there exists an increasing sequence of measurable subsets M_k with $\mu(M_k) < \infty$ for every k and $\bigcup_k M_k = M$. We say that a point x *goes to infinity* if, for every k, there exists only a finite number of iterates of x that are in M_k. Show that, given any $E \subset M$ with positive measure, μ-almost every point $x \in E$ returns to E an infinite number of times or else goes to infinity.

1.2.4. Let $f : M \to M$ be a measurable transformation, not necessarily invertible, μ be an invariant probability measure and $D \subset M$ be a set with positive measure. Prove that almost every point of D spends a *positive fraction of time* in D:

$$\limsup_n \frac{1}{n}\#\{0 \leq j \leq n-1 : f^j(x) \in D\} > 0$$

for μ-almost every $x \in D$. [Note: One may replace lim sup by lim inf in the statement, but the proof of that fact will have to wait until Chapter 3.]

1.2.5. Let $f : M \to M$ be a measurable transformation preserving a finite measure μ. Given any measurable set $A \subset M$ with $\mu(A) > 0$, let $n_1 < n_2 < \cdots$ be the sequence of values of n such that $\mu(f^{-n}(A) \cap A) > 0$. The goal of this exercise is to prove that $V_A = \{n_1, n_2, \dots\}$ is a *syndetic*, that is, that there exists $C > 0$ such that $n_{i+1} - n_i \leq C$ for every i.

(a) Show that for any increasing sequence $k_1 < k_2 < \cdots$ there exist $j > i \geq 1$ such that $\mu(A \cap f^{-(k_j - k_i)}(A)) > 0$.

(b) Given any infinite sequence $\ell = (l_j)_j$ of natural numbers, denote by $S(\ell)$ the set of all finite sums of consecutive elements of ℓ. Show that V_A intersects $S(\ell)$ for every ℓ.

(c) Deduce that the set V_A is syndetic.

[Note: Exercise 3.1.2 provides a different proof of this fact.]

1.2.6. Show that if $f : [0,1] \to [0,1]$ is a measurable transformation preserving the Lebesgue measure m then m-almost every point $x \in [0,1]$ satisfies

$$\liminf_n n|f^n(x) - x| \leq 1.$$

[Note: Boshernitzan [Bos93] proved a much more general result, namely that $\liminf_n n^{1/d} d(f^n(x), x) < \infty$ for μ-almost every point and every probability measure μ invariant under $f : M \to M$, assuming M is a separable metric whose d-dimensional Hausdorff measure is σ-finite.]

1.2.7. Define $f : [0,1] \to [0,1]$ by $f(x) = (x + \omega) - [x + \omega]$, where ω represents the *golden ratio* $(1 + \sqrt{5})/2$. Given $x \in [0,1]$, check that $n|f^n(x) - x| = n^2|\omega - q_n|$ for every n, where $(q_n)_n \to \omega$ is the sequence of rational numbers given by $q_n = [x + n\omega]/n$. Using that the roots of the polynomial $R(z) = z^2 - z - 1$ are precisely ω and $\omega - \sqrt{5}$, prove that $\liminf_n n^2|\omega - q_n| \geq 1/\sqrt{5}$. [Note: This shows that the constant 1 in Exercise 1.2.6 cannot be replaced by any constant smaller than

$1/\sqrt{5}$. It is not known whether 1 is the smallest constant such that the statement holds for *every* transformation on the interval.]

1.3 Examples

Next, we describe some simple examples of invariant measures for transformations and flows that help us interpret the significance of the Poincaré recurrence theorems and also lead to some interesting conclusions.

1.3.1 Decimal expansion

Our first example is the transformation defined on the interval $[0,1]$ in the following way:

$$f : [0,1] \to [0,1], \quad f(x) = 10x - [10x].$$

Here and in what follows, we use $[y]$ as the *integer part* of a real number y, that is, the largest integer smaller than or equal y. So, f is the map sending each $x \in [0,1]$ to the *fractional part* of $10x$. Figure 1.1 represents the graph of f.

We claim that the Lebesgue measure μ on the interval is invariant under the transformation f, that is, it satisfies

$$\mu(E) = \mu(f^{-1}(E)) \quad \text{for every measurable set } E \subset M. \tag{1.3.1}$$

This can be checked as follows. Let us begin by supposing that E is an interval. Then, as illustrated in Figure 1.1, its pre-image $f^{-1}(E)$ consists of ten intervals, each of which is ten times shorter than E. Hence, the Lebesgue measure of $f^{-1}(E)$ is equal to the Lebesgue measure of E. This proves that (1.3.1) does hold in the case of intervals. As a consequence, it also holds when E is a finite

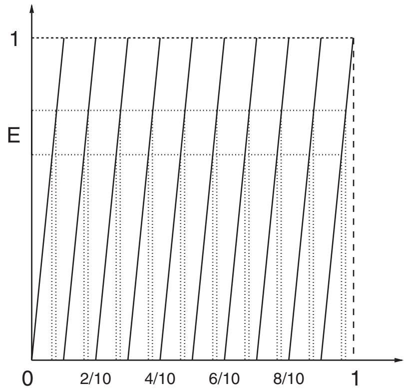

Figure 1.1. Fractional part of $10x$