

---

## Index

- acceleration, xii
- action functional, 173
  - and Newton's equation (Maupertuis's principle), 173
- action of a group on a set, 206
  - transitive, 157
- affine part of a subset of  $\mathbb{RP}^2$ , 143
- algebraic multiplicity of an eigenvalue, 185
- angle between tangent vectors, 120
- angle sum
  - in a hyperbolic triangle, 159
  - in a spherical triangle, 157
- angular momentum, 2, 64
  - and curvature, 138
  - is a constant of motion in the  $n$ -body problem, 64
  - is a constant of motion in the CFP, 2
- angular velocity, 4
  - instantaneous, 80, 96
- anomaly
  - eccentric, 37, 42
  - integral description, 47
  - mean, 38
  - relation between the true and eccentric – in the elliptic case, 46
  - true, 28
- Apollonius of Perga (third century BC), 21
- arc length, 115, 135
  - parametrisation, 116, 136
  - length functional w.r.t. a Riemannian metric, 171
- area element in polar coordinates, 5
- area in the hyperbolic plane, 158
- area swept out by the position vector, 4, 9
- argument of the pericentre, 50
- asymptote of a hyperbola, 22
- autonomous
  - differential equation, 86
  - Hamiltonian system, 178
  - Lagrangian function, 171
- barycentre, *see* centre of mass
- boundary at infinity, 121
- branch of a hyperbola, 15
- bump function, 170
- canonical system, *see* Hamilton equations
- canonical transformation, 180
  - also called 'symplectomorphism', 182
- Cardano, Gerolamo (1501–1576), 45
  - formula for cubic equations, 43
- Cauchy–Lipschitz theorem, xiv
- central configuration, 67
  - as a critical point condition, 68
  - finiteness question, 72
  - for four bodies, 69
  - of equal mass, 72
  - for three bodies, 69
  - gives rise to a homothetic solution of the  $n$ -body problem, 67
  - planar, 69
  - planar – gives rise to a homographic solution of the  $n$ -body problem, 69
  - planar – gives rise to a relative equilibrium, 71
  - regular  $n$ -gon is a – for  $n$  equal masses, 69
- central force problem, 1
  - as a first-order system, 2
  - centrally symmetric, 7
  - is conservative, 7
  - motion is planar, 2
  - symmetries, 7

- centre of mass
  - of a two-body system, 52
  - of an  $n$ -body system, 64
- (CFP), *see* central force problem
- characteristic polynomial, 185
- Cherry's example concerning stability, 191
- choreographic solution of the  $n$ -body problem, 93
- circles are characterised by constant curvature, 137
- compact topological space, 153
- cone, 20, 58
  - complement of the thick diagonal is a –, 58
- configuration space, 178
  - for (PCR3B), 178
- conformal map, 110
  - and the Jacobian matrix, 153
- conformally equivalent Riemannian metrics, 121
  - Jacobi metric and euclidean metric, 173
- conic section, 15, 20, 21
  - as a solution of the Kepler problem, 24
- conservative force field, 6
  - centrally symmetric CFP, 7
  - Kepler problem, 27
  - $n$ -body problem, 56
- constant of motion, 2
  - angular momentum, 2, 64
  - autonomous Hamiltonian, 178, 181
  - energy, 6, 57
  - for the Hooke problem, 147
  - for the Kepler problem (orbital elements), 49
  - for the  $n$ -body problem, 64
  - for the two-body problem, 52
  - from an autonomous Lagrangian function, 172
  - geometric interpretation, 88
  - Jacobi constant, 87
  - linear momentum, 64
- continuous map between topological spaces, 153
- convex hull, 73
- convex subset of  $\mathbb{R}^d$ , 73
- coordinates
  - homogeneous, 142
  - Jacobi –, 75
  - 'no set of – is good enough', 72
  - spherical, 154
- co-planar motion, 69
- covering map, 4
- cubic equation
  - general form, 47
  - in the parabolic case of the Kepler problem, 42
  - solution by radicals (Cardano's formula), 43–45
- curtate cycloid, 40
- curvature of a curve on a surface in  $\mathbb{R}^3$ 
  - geodesic, 145
  - normal, 145
- curvature of a planar curve, 137
  - and angular momentum of its polar reciprocal, 138, 160
  - and central force, 165
  - and the duality of force laws, 165
  - constant curvature characterises circles, 137
  - transformation under a holomorphic map, 161
- cycloid, 40
  - curtate, 40
  - describes solutions of (K) with  $c = 0$  and  $h < 0$ , 47
- Descartes's sign rule for polynomials, 98
- diagonal
  - in  $\mathbb{R}^3 \times \mathbb{R}^3$ , 51
  - 'thick' – in  $\mathbb{R}^{3n}$ , 55
- differential of a  $C^1$ -map
  - geometric interpretation, 120
  - on  $S^2$ , 207
  - on hyperbolic space, 120
- directrix
  - of ellipse, 13
    - role in Newton's geometric solution of the Kepler equation, 40
  - of hyperbola, 18
  - of parabola, 18
- dual of a curve in  $\mathbb{R}P^2$ , 144
  - and polar reciprocal, 145
- dual projective plane, 143
- duality between conics and circles, 138
- duality between poles and polars, 134
  - and polar reciprocation, 138, 159
  - projective version, 144
- duality between solutions of (H) and (K), 147
  - via holomorphic transformations, 175
  - was found by Newton, 151
- duality of force laws, 164, 174
  - and curvature of planar curves, 165
- eccentric anomaly
  - and the Levi-Civita parameter, 131
  - elliptic case, 37

- gives the arc length parametrisation of the transformed hodograph, 117, 127, 128
- hyperbolic case, 47
- integral description, 47, 128, 130, 159
- parabolic case, 42
- eccentricity (vector)
  - of ellipse, 11
    - as one of the six orbital elements, 50
  - of hyperbola, 15
  - of parabola, 18
- ecliptic, 49
- eigenvalue
  - algebraic multiplicity, 185
  - geometric multiplicity, 184
- ellipse, 10
  - as a solution of the Hooke problem, 147
  - as a solution of the Kepler problem, 24
  - centre, 10
  - description
    - as a conic section, 21
    - by a quadratic equation, 14
    - by a vector equation, 11
    - in polar coordinates, 12, 20
    - via the gardener's construction, 10
    - with a directrix, 13, 20
  - directrix, 13
  - eccentricity (vector), 11
  - foci, 10
  - parametrisation in terms of the eccentric anomaly, 38
  - pericentre, 37
  - pericentre distance, 12
  - reflection property, 35
  - semi-major axis, 10
  - semi-minor axis, 13
- energy
  - and the Euler–Lagrange equation, 172
  - in the Hooke problem, 147
  - in the Kepler problem, 27
  - is a constant of motion in a conservative force field, 6
  - kinetic, 6
    - of an  $n$ -body system, 57
  - total (kinetic + potential), 6
    - as a Hamiltonian function, 178
    - in the  $n$ -body problem, 57
- envelope of a set of lines, 161
- equations
  - (CFP): central force problem, 1
  - (CFP'): (CFP) as a first-order system, 2
  - (EL): Euler–Lagrange equation, 168
  - (EL'): (EL) for an autonomous Lagrangian function, 171
  - (H): Hooke problem, 146
    - solutions of type  $(H_0)$ ,  $(H_{\pm})$ , 146
  - Hamiltonian system, 177
    - as a flow equation, 181
    - in complex notation, 179
  - (K): Kepler problem, 23
  - (K'): (K) as a first-order system, 25
  - (K) in Hamiltonian form, 197
  - (N): Newton's equation of motion, 6
  - $(N_c)$ : (N) for a conservative force field, 6, 173
  - $n$ -body problem, 55
    - as a conservative system, 57
    - as a first-order system, 59
    - central configuration, 67
    - homothetic solution, 67
  - (PCR3B): planar circular restricted three-body problem in rotating coordinates, 86
  - (PCR3B'): (PCR3B) as a first-order system, 87
  - (PCR3B) in Hamiltonian form, 166–167
  - (R3B): restricted three-body problem, 84
  - three-body problem
    - collinear circular solution, 83
    - in Jacobi coordinates, 75
  - two-body problem, 51
    - in barycentric coordinates, 53
    - in relative coordinates, 52
- equilibrium point
  - asymptotic stability, 185
  - of a dynamical system, 185
  - infinitesimal stability, 185
  - of a general first-order system
    - Lyapunov vs. infinitesimal stability, 185
  - of a Hamiltonian system, 182
  - infinitesimal stability, 187
  - stability of the origin in a linear system, 185
- equilibrium solution, 58
  - $n$ -body problem does not have any, 58
  - of (PCR3B), *see* libration points
- Euler points, 90
  - are not stable, 193
  - are saddle points of  $\Phi$ , 92
- Euler, Leonhard (1707–1783)
  - collinear solutions of the three-body problem, 83–84, 99–100
  - theorem on positive homogeneous functions, 58

- Euler, Leonhard (*cont.*)  
 work on the three-body problem, 93–94
- Euler–Lagrange equation, 168  
 and Newton’s equation (Hamilton’s principle), 169  
 for an autonomous Lagrangian function, 171
- existence and uniqueness theorem for  
 first-order differential equations  
 (Picard–Lindelöf), xiv  
 existence of solutions for all times, 25
- fall time, 48
- flow line of a vector field, xiv
- focus  
 of ellipse, 10  
 of hyperbola, 15  
 of parabola, 18
- force field, 6  
 conservative, 6  
 for the centrally symmetric CFP is  
 conservative, 7  
 potential of, 6
- fractional linear transformation, 157  
 is an isometry of the hyperbolic plane, 157
- fundamental lemma of the calculus of  
 variations, 169
- Galilean relativity principle, 51
- gardener’s construction  
 of ellipse, 10  
 of hyperbola, 15  
 of parabola, 18
- general linear group, 189
- generalised momenta, 178
- geodesic, 115  
 for the Jacobi metric, 173  
 hyperbolic, *see* hyperbolic geodesic  
 in a subset of  $\mathbb{R}^n$  with a Riemannian metric,  
 171  
 on  $S^2$ , 155, 171  
 on  $S^3$ , 115  
 on  $S^{n-1}$ , 156  
 on a submanifold in  $\mathbb{R}^n$   
 as a locally distance minimising curve,  
 154  
 characterised by the acceleration vector,  
 155  
 on a surface in  $\mathbb{R}^3$ , 146  
 variational characterisation, 156, 171
- geodesic curvature, 145
- geodesic flow on  $S^2$ , 195–197
- geometric multiplicity of an eigenvalue, 185
- gravitational constant  $G$ , xiii
- Graßmann identity for the vector product, 3, 8
- great circle, 115
- (H), *see* Hooke problem
- $H$  is for Huygens, 187
- Hamilton equations, 167, 177  
 as a flow equation, 181  
 in complex notation, 179  
 linear approximation at an equilibrium  
 point, 186
- Hamilton’s principle, 169
- Hamilton, William Rowan (1805–1865)  
 discovery of the quaternions, 200  
 hodograph theorem, 102, 162  
 invention of the hodograph, 150
- Hamiltonian function, 167  
 derived from a Lagrangian function  
 (Legendre transformation), 177  
 for motions in a conservative force field  
 (energy), 178  
 $H$  stands for Huygens, 187
- Hamiltonian system, 178  
 autonomous, 178  
 has  $H$  as a constant of motion, 178, 181  
 configuration space, 178  
 equilibrium point, 182  
 generalised momenta, 178  
 phase portrait, 190  
 phase space, 178
- Hamiltonian vector field, 181
- Hill’s region, 91, 172
- hodograph, 102  
 circularity of the – characterises the  
 Newtonian law of attraction, 162  
 Hamilton’s theorem, 102  
 Moser’s theorem, 115  
 Moser–Osipov–Belbruno theorem, 131  
 of a regularised collision solution, 114  
 theorem of Osipov and Belbruno, 126
- homeomorphism, xiv, 153
- homogeneous coordinates for projective space,  
 142
- homographic solution of the  $n$ -body problem,  
 69  
 is homothetic iff  $c = 0$ , 97  
 planar – comes from a central configuration,  
 69
- homographic solution of the three-body  
 problem (Lagrange’s theorem), 77  
 circular case, 94  
 Laplace’s proof, 93

- homothetic solution of the  $n$ -body problem, 67  
 homothety, 67  
 Hooke problem  
   duality with the Kepler problem, *see* duality  
     between solutions of (H) and (K)  
   is conservative, 147  
   solutions, 146  
 Hooke, Robert (1635–1703)  
   role in the discovery of the inverse square  
   law, 23, 31  
 Huygens, Christiaan (1629–1695)  
   formulation of energy conservation, 187  
 hyperbola, 15  
   as a solution of the Hooke problem, 147  
   as a solution of the Kepler problem, 24  
   asymptote, 22  
   branches, 15  
   description  
     by a quadratic equation, 21  
     by a vector equation, 15  
     in polar coordinates, 17, 20  
     via the gardener's construction, 15  
     with a directrix, 17, 20  
   directrix, 18  
   eccentricity (vector), 15  
   foci, 15  
   parametrisation in terms of the eccentric  
   anomaly, 22, 46  
   pericentre, 38  
   pericentre distance, 17  
   principal branch, 16  
   real semi-axis, 15  
   reflection property, 35  
 hyperbolic area, 158  
 hyperbolic geodesic, 118  
   in the half-space model, 119  
   via variational principle, 172, 188  
 hyperbolic length, 118  
 hyperbolic line, 122  
   in the half-space model, 122  
   in the Poincaré disc model, 125  
 hyperbolic space  
   boundary at infinity, 121  
   half-space model, 120  
   hyperbolic  $k$ -plane in  $\mathbb{H}^3$ , 158  
   Poincaré disc model, 125  
 hyperbolic triangle, 158  
 inclination, 49  
 instantaneous angular velocity, 80, 96  
 integral curve of a vector field, xiv  
 intrinsic normal vector, 145  
 inversion, 108  
   and polar reciprocation, 133  
   as 'Wiedergeburt und Auferstehung', 150  
   fixed point set is the sphere of inversion, 109  
   is an involution, 108  
   is conformal, 110, 150  
   sends spheres and planes to spheres and  
   planes, 110  
   Steiner's theorem, 110  
   yields hyperbolic isometries, 121  
 involution, 108  
 isometry, 51  
   of  $\mathbb{H}^3$ , 120, 157  
 isometry group, 157  
   of  $\mathbb{H}^3$  acts transitively, 157  
   of the hyperbolic plane, 158  
 Jacobi constant, 87  
 Jacobi coordinates, 75  
 Jacobi integral, 87  
   as a Hamiltonian function for (PCR3B), 167  
   defines three-dimensional submanifolds, 88  
 Jacobi metric, 173  
   geodesics are solutions of  $(N_c)$ , 173  
 Jacobi, Carl Gustav Jacob (1804–1851)  
   introduction of the Jacobi metric, 187  
 (K), *see* Kepler problem  
 Kepler equation  
   cubic analogue in the parabolic case, 42  
   for elliptic solutions, 38  
   for hyperbolic solutions, 47  
   solution by Newton's iterative method, 45  
   solution by the cycloid, 40–41  
   solution in terms of Bessel functions, 45  
 Kepler problem, 23  
   as a first-order system, 25  
   as a Hamiltonian system, 197  
   constants of motion (orbital elements), 49  
   duality with the Hooke problem, *see* duality  
     between solutions of (H) and (K)  
   energy, 27  
   hodograph, *see* hodograph, Moser, and  
     Osipov–Belbruno theorem  
   is conservative, 7, 27  
   is of order six, 49  
   regularisation, *see* regularisation of  
     collisions  
   solution with  $c = 0$  and  $h = 0$ , 49  
   solutions with  $c = 0$  and  $h < 0$ , 48  
   solutions with  $c \neq 0$  (Kepler's first law), 24  
   solutions with  $c \neq 0$  are defined for all  
   times, 25

- Kepler's first law, 24
  - Lagrange's proof, 32
  - Laplace's proof, 24, 30
  - proof by van Haandel and Heckman, 33
  - proof via a differential equation on the inverse radius, 31
  - proof via hodograph, 104
  - proof via Newton–Hooke duality, 149, 163
  - proof via polar reciprocation, 139
  - was proved by Newton, 30
- Kepler's second law, 5
  - converse, 6
  - was proved by Newton, 7
- Kepler's third law, 29, 39
  - for circular motions, 9
  - for several planets, 53
  - was proved by Newton, 30
- kinetic energy, 6
  - of an  $n$ -body system, 57
- Lagrange points, 90
  - are minima of  $\Phi$ , 91, 100
  - condition for stability, 193
  - Trojan asteroids, 93
- Lagrange, Joseph-Louis (1736–1813)
  - proof of Kepler's first law, 32
  - theorem on homographic solutions of the three-body problem, 77
    - circular case, 94
    - Laplace's proof, 93
- Lagrange–Jacobi identity, 63
  - virial theorem, 71
- Lagrangian function, 168
  - for motions in a conservative force field (kinetic minus potential energy), 169
- Lambert's theorem, 45
- Laplace, Pierre-Simon (1749–1827)
  - proof of Kepler's first law, 30
  - proof of Lagrange's theorem on the three-body problem, 93
- Laplace–Runge–Lenz vector, 25
- Legendre condition
  - for the Hamiltonian function, 178
  - for the Lagrangian function, 176
- Legendre transformation, 178
- length functional, 171
- Levi-Civita parameter, 131
  - and regularisation, 150
  - and the eccentric anomaly, 131
- libration points, 90
  - Euler points, 90
    - are not stable, 193
  - Lagrange points, 90
    - condition for stability, 193
  - stability, 191–193
- linear momentum, 64
  - is a constant of motion in the  $n$ -body problem, 64
- longitude of the ascending node, 49
- Lorentz group, 206
- Möbius transformation, 157
  - is an isometry of the hyperbolic plane, 157
- Maupertuis's principle, 173
- mean anomaly, 38
- moment of inertia, 62
  - computed from pairwise distances, 73
  - goes to  $\infty$  for  $n$ -body systems with  $h > 0$  and  $\omega = \infty$ , 63
- Lagrange–Jacobi identity, 63
- Sundman's inequality, 65, 73–74
- momentum
  - angular, *see* angular momentum
  - generalised momenta, 178
  - linear, *see* linear momentum
- Moser, Jürgen (1928–1999)
  - theorem on hodographs, 115
    - general version, 131
    - parametric version, 117
- $n$ -body problem, 55
  - as a first-order system, 59
  - central configuration, 67
  - choreographic solution, 93
  - constants of motion, 64
  - co-planar solution, 69
  - does not have any equilibrium solutions, 58
  - energy, 57
  - homographic solution, 69
  - homothetic solution, 67
  - is conservative, 56
  - planar solution, 69
  - relative equilibrium, 70
  - solutions defined in finite time only, 59, 71
  - total collapse, 65
- Newton potential, 56
  - is positive homogeneous of degree  $-1$ , 58
- Newton's equation of motion, 6
  - in a conservative force field, 6, 173
  - solutions are geodesics for the Jacobi metric, 173
- Newton, Isaac (1643–1727)
  - curvature of planar curves and central force, 165
  - duality of force laws, 151

- Newton's iterative method, 45  
 proof of Kepler's first law, 30  
 proof of Kepler's second law, 7  
 proof of Kepler's third law, 30  
 solution of the Kepler equation, 40, 45
- Newtonian law  
   of gravitation, xiii, 23  
   second – of motion, xiii
- non-autonomous differential equation, 85
- normal curvature, 145
- one-point compactification, 152
- orbital elements, 49
- order  
   of (K) is six, 49  
   of (PCR3B) is three, 86  
   of (R3B) is six, 86  
   of the planar restricted three-body problem  
     is four, 86  
   of the three-body problem with fixed centre  
     of mass is twelve, 75  
   of the two-body problem is twelve, 52
- Osipov–Belbruno theorem, 126  
   general version, 131  
   parametric version  
     hyperbolic case, 128  
     parabolic case, 127
- parabola, 18  
   as a solution of the Kepler problem, 24  
   description  
     as a conic section, 20  
     by a quadratic equation, 20  
     by a vector equation, 18  
     in polar coordinates, 19, 20  
     via the gardener's construction, 18  
     with a directrix, 18, 20  
   directrix, 18  
   focus, 18  
   parametrisation in terms of the eccentric  
     anomaly, 42  
   pericentre, 38  
   reflection property, 35, 162
- (PCR3B), *see* planar circular restricted  
   three-body problem
- pericentre, 37  
   argument of the  $\omega$ , 50  
   called 'perihelion' for the Sun being the  
     central body, 50
- pericentre passage, 38  
   as one of the six orbital elements, 50
- perihelion, 50
- period, 29, 33
- phase portrait of a Hamiltonian system, 190
- phase space, 178
- Picard–Lindelöf theorem, xiv
- planar central configuration, 69
- planar circular restricted three-body problem,  
   85  
   as a first-order system, 87  
   as a Hamiltonian system, 166–167  
   Euler points, 90  
   Hill's region, 91  
   in rotating coordinates, 86  
   is a model for the Trojan asteroids, 86  
   is of order three, 86  
   Jacobi constant, 87  
   Jacobi integral, 87  
     as a Hamiltonian function, 167  
   Lagrange points, 90  
   libration points, 90  
   zero velocity curves, 91
- planar motion, 69
- plane  
   hyperbolic – in  $\mathbb{H}$ , 158  
    $k$ -dimensional – in  $\mathbb{R}^n$ , 110  
   radial, 102
- points at infinity in projective space, 142
- polar, 133
- polar reciprocal, 135  
   defined projectively, 145  
   duality, 137  
     of conics and circles, 138, 162  
     projective version, 145  
   in cartesian coordinates, 135  
   is the envelope of the family of polars, 161  
   relation with the velocity curve, 140
- polarisation identity, 122
- pole, 133
- positive homogeneous function, 58  
   Euler's theorem, 58  
   Newton potential is a  $-$ , 58
- potential  
   Newton – for the  $n$ -body problem, 56  
   of a centrally symmetric CFP, 7  
   of a conservative force field, 6  
   of the Newtonian central force, 7
- primaries in the restricted three-body problem,  
   85
- principal branch of a hyperbola, 16
- projective line, 142  
   is diffeomorphic to a circle, 206  
   is homeomorphic to a circle, 163

- projective plane, 141
  - affine part of a subset, 143
  - dual, 143
  - line in the –, 142
  - points at infinity, 142
- projective space, 142
  - points at infinity, 142
  - $\mathbb{R}P^3$  is diffeomorphic to  $SO(3)$ , 204–205
  - $\mathbb{R}P^3$  is homeomorphic to  $SO(3)$ , 199–200, 202
- quaternions, 200
  - conjugation, 201
  - form a division algebra, 201
  - imaginary part, 201
  - norm, 201
  - real part, 201
  - relation with the inner and cross product, 202
  - unit –, 202
  - were discovered by Hamilton, 200
- quotient topology, 141
- (R3B), *see* restricted three-body problem
- radial plane, 102
- real semi-axis of a hyperbola, 15
- reflection of light rays in a mirror, 34
- reflection property of conics, 35, 162
- regular  $C^1$ -curve, 118, 135
  - in  $\mathbb{R}P^2$ , 144
  - dual curve, 144
- regularisation of collisions, 48
  - and an integral description of the eccentric anomaly, 159
  - and Newton–Hooke duality, 149
  - and the hodograph, 114
  - and the Levi-Civita parameter, 150
  - in Moser’s theorem, 117
  - in the Osipov–Belbruno theorem, 126–128
- relation between
  - $a$  and  $h$ , 28
  - $a$ ,  $c$  and  $e$ , 25
  - $c$ ,  $e$  and  $h$ , 27
- relative equilibrium, 70
  - always comes from a planar central configuration, 71
  - in the three-body problem, 78–79
  - is planar, 97
  - rotates with constant angular velocity, 97
- restricted three-body problem, 84
  - as a non-autonomous differential equation, 85
  - conservation laws do not hold, 86
  - is of order six, 86
  - planar – is of order four, 86
  - primaries, 85
- Riemannian metric, 119
  - and length functional, 171
  - conformal equivalence, 121
  - induced on a submanifold, 130, 155
  - on an open subset of  $\mathbb{R}^n$ , 171
- semi-major axis of an ellipse, 10
  - as one of the six orbital elements, 50
- semi-minor axis of an ellipse, 13
- smooth map, xiii
- $SO(3)$ 
  - elements are rotations, 200
  - is a three-dimensional manifold, 198
  - is diffeomorphic to  $\mathbb{R}P^3$ , 204–205
  - is diffeomorphic to the unit tangent bundle of  $S^2$ , 199, 207
  - is homeomorphic to  $\mathbb{R}P^3$ , 199–200, 202
  - solutions of a linear differential equation, 184
- Somerville, Mary (1780–1872), 30
- space forms in dimension three, 131
- speed, xii
- sphere
  - $k$ -dimensional – in  $\mathbb{R}^n$ , 110, 152
  - unit – in  $\mathbb{R}^n$ , 107
- spherical coordinates, 154
- spherical triangle, 157
- stability
  - asymptotic, 185
  - Cherry’s example, 191
  - infinitesimal, 185
    - condition in terms of the characteristic roots, 187
  - infinitesimal – does not imply Lyapunov –, 190–191
  - of the libration points, 191–193
  - of the origin in a linear system, 185
  - relation between infinitesimal – and Lyapunov –, 185
- Steiner, Jakob (1796–1863)
  - theorem on inversions, 110, 150
- stereographic projection, 109
  - coordinate description, 113
  - is an inversion, 109
  - is conformal, 111, 153
  - sends circles to circles or lines, 111
- Sundman, Karl (1873–1949)
  - inequality, 65, 73–74
  - case of equality, 74

- power series solution to the three-body problem, 71
- theorem on solutions to the  $n$ -body problem
  - defined in finite time only, 71
  - theorem on total collapse, 66
- symplectic form
  - canonical – on  $\mathbb{R}^{2n}$ , 180
  - on a manifold, 182
- symplectic group, 188
  - relation with the unitary group, 189
- symplectic matrix, 180
  - condition in terms of a block decomposition, 188
- symplectomorphism, 182
- tangent line of a curve in  $\mathbb{R}P^2$ , 144
- tangent space
  - at a point in  $\mathbb{R}^n$ , 171
  - of  $SO(3)$  at  $E$ , 205
  - of  $S^2$ , 196
  - of a submanifold in  $\mathbb{R}^n$ , 155
  - of hyperbolic space, 119
- three-body problem
  - Euler's collinear solutions, 83–84, 99–100
  - figure-eight solution, 93
  - in Jacobi coordinates, 75
  - Lagrange's homographic solutions, 77–83
    - circular case, 94
    - Laplace's proof, 93
  - planar circular restricted, *see* planar circular restricted three-body problem
  - relative equilibrium, 78–79
  - restricted, *see* restricted three-body problem
  - solutions with  $c = 0$  are planar, 75
- topology, 152
  - compactness, 153
  - continuity, 153
  - induced on a subset, 152
  - on a one-point compactification, 152
  - quotient –, 141
- total collapse, 65
  - can happen only in *planar* three-body systems, 75
  - can happen only in finite time, 65
  - can happen only in systems with  $c = 0$  (Sundman's theorem), 66
- transitive group action, 157
- triangle
  - hyperbolic, 158
  - spherical, 157
- Trojan asteroids, 78
  - are modelled by (PCR3B), 86
- true anomaly, 28
- two-body problem, 51
  - circular solution, 53
  - constants of motion, 52, 64
  - in barycentric coordinates, 53
  - in relative coordinates, 52
  - invariance properties, 51
  - is of order twelve, 52
- unit speed curve, *see* arc length parametrisation
- unit tangent bundle of  $S^2$ , 196
  - is a three-dimensional manifold, 196
  - is diffeomorphic to  $SO(3)$ , 199, 207
- unitary group, 189
  - relation with the symplectic group, 189
- unitary matrix, 189
- variational principles
  - characterisation of geodesics, 156, 171
  - Hamilton's principle, 169
  - Maupertuis's principle, 173
- vector field, xiv
  - and first-order differential equation, xiv
  - integral curve (or flow line) of a –, xiv
- velocity, xii, 2
  - angular, *see* angular velocity
- velocity circle, 104
- velocity curve, 102
  - relation with the polar reciprocal, 140
- vernal equinox, 49
- virial theorem, 71
- volume of a parallelepiped, 31
- wedge product of two 1-forms, 180
- zero velocity curves, 91