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The Geometry of Celestial Mechanics

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Und in der Tat, die wichtigsten geistigen Vorkehrungen der Menschheit dienen der Erhaltung eines beständigen Gemütszustands, und alle Gefühle, alle Leidenschaften der Welt sind ein Nichts gegenüber der ungeheuren, aber völlig unbewußten Anstrengung, welche die Menschheit macht, um sich ihre gehobene Gemütsruhe zu bewahren! Es lohnt sich scheinbar kaum, davon zu reden, so klaglos wirkt es. Aber wenn man näher hinsieht, ist es doch ein äußerst künstlicher Bewußtseinszustand, der dem Menschen den aufrechten Gang zwischen kreisenden Gestirnen verleiht und ihm erlaubt, inmitten der fast unendlichen Unbekanntheit der Welt würdevoll die Hand zwischen den zweiten und dritten Rockknopf zu stecken.

Robert Musil, *Der Mann ohne Eigenschaften*

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Preface

DER INQUISITOR Und da richten diese Würmer von Mathematikern ihre Rohre auf den Himmel [...] Ist es nicht gleichgültig, wie diese Kugeln sich drehen?

Bertolt Brecht, *Leben des Galilei*

Celestial mechanics has attracted the interest of some of the greatest mathematical minds in history, from the ancient Greeks to the present day. Isaac Newton's deduction of the universal law of gravitation (Newton, 1687) triggered enormous advances in mathematical astronomy, spearheaded by the mathematical giant Leonhard Euler (1707–1783). Other mathematicians who drove the development of celestial mechanics in the first half of the eighteenth century were Alexis Claude Clairaut (1713–1765) and Jean le Rond d'Alembert (1717–1783), see (Linton, 2004). In those days, the demarcation lines separating mathematics and physics from each other and from intellectual life in general had not yet been drawn. Indeed, d'Alembert may be more famous as the co-editor with Denis Diderot of the *Encyclopédie*. During the Enlightenment, celestial mechanics was a subject discussed in the salons by writers, philosophers and intellectuals like Voltaire (1694–1778) and Émilie du Châtelet (1706–1749).

The history of celestial mechanics continues with Joseph-Louis Lagrange (1736–1813), Pierre-Simon de Laplace (1749–1827) and William Rowan Hamilton (1805–1865), to name but three mathematicians whose contributions will be discussed at length in this text. Henri Poincaré (1854–1912), perhaps the last universal mathematician, initiated the modern study of the three-body problem, together with large parts of the theory of dynamical systems and what is now known as symplectic geometry (Barrow-Green, 1997; Charpentier *et al.*, 2010; McDuff and Salamon, 1998).

Yet this time-honoured subject seems to have all but vanished from the mathematical curricula of our universities. This is reflected in the available textbooks, which are either getting a bit long in the tooth, or are addressed to a fairly advanced and specialised audience. The *Lectures on Celestial Mechanics* by Siegel and Moser (1971), a classic in their own right, deal with Sundman's work on the three-body problem in the wake of Poincaré's, and with questions about periodic solutions and stability, all at a rather mature level. Celestial mechanics as a key motivation for the study of dynamical systems is served well by (Moser and Zehnder, 2005) and (Meyer *et al.*, 2009).

My personal interest in celestial mechanics stems from reading the paper (Albers *et al.*, 2012), where the three-body problem is approached with methods from contact topology, my core area of expertise, see (Geiges, 2008). I should say 'attempting to read', for I quickly realised that I was ignorant of some of the most basic terminology in celestial mechanics.

In order to remedy this deplorable state of affairs – and to confute the inquisitor – I decided to teach a course on celestial mechanics, with (Pollard, 1966), (Danby, 1992) and (Ortega and Ureña, 2010) as my excellent guides. The latter textbook can be recommended even to readers whose grasp of Spanish is as rudimentary as mine.

However, none of these texts takes the geometric view that I wished to emphasise, so I included material from sources such as (Milnor, 1983) and (Hall and Josić, 2000), expanded and adapted to the needs of an introductory course. The present text rather faithfully reflects the course I taught at the University of Cologne in 2012/13, where the audience of some seventy ranged from second-year mathematics or physics undergraduates all the way to Ph.D. students. For a follow-up seminar in 2014/15 and this write-up I added more geometric material, notably on the curvature of planar curves and projective geometry, inspired by (Coolidge, 1920), and I removed a couple of sections on generating functions and Hamilton–Jacobi theory, which I felt were less in the spirit of this elementary geometry course in disguise.

The result, I hope, is a text that can be read profitably by undergraduates in their penultimate or final year, while not being too pedestrian for more advanced students. I believe that, for students not intending to specialise in geometry, learning elementary differential geometry and topology by seeing it 'in action', that is, applied to questions in celestial mechanics, may be a more satisfying experience than some traditional courses that concentrate on the development of machinery and often stop before the student can really appreciate its utility – needless to say, students who plan to continue with further courses in geometry may likewise enjoy that experience. Celestial mechanics is a field where many strands of pure and applied mathematics come together, and for this reason alone it deserves a more prominent place in the curriculum.

I have included over a hundred exercises, often with comments that explain their relevance, making the text suitable for self-study. It should be possible to cover most of this book in a one-semester course of 14 weeks. For shorter courses one could omit the proof of planarity in Lagrange's theorem (Theorem 7.1) and make a selective choice of the material in Chapters 8 to 10.

The contents of this book

A large portion of this text is concerned with the simplest question in celestial mechanics, the Kepler problem, which studies the motion of a single body around a fixed centre under Newtonian attraction. One of my aims is to display the rich geometry of this problem. In particular, several proofs of Kepler's first law about the shape of the orbit will be given, based on geometric concepts such as curvature of planar curves or conformal (i.e. angle-preserving) transformations of the plane.

Chapter 1 introduces the central force problem, where the force law need not be Newtonian. Even in this more general setting one finds two preserved quantities of the motion: the angular momentum and, if the force field derives from a potential, the energy. The preservation of the angular momentum can be rephrased as Kepler's second law about areas.

Kepler's first law about the shape of the orbit, now assuming Newtonian attraction, is proved (following Laplace) in Chapter 3: the orbit is a conic section, with one focus in the force centre. Chapter 2 provides the background on conic sections, to which the reader may refer as needed.

Of course, knowing the shape of the orbit is only half the answer, in particular if you are trying to locate a celestial object in the sky. One would really like to have an explicit time parametrisation of the orbit. This surprisingly difficult question is the theme of Chapter 4. In the elliptic and hyperbolic case it leads to a transcendental equation named after Kepler; I present a geometric solution of this equation, due to Newton, involving a famous planar curve, the cycloid. In the parabolic case it leads to a cubic equation, and I reveal the geometry behind the algebraic solution of such equations.

Passing from one to two bodies moving under mutual attraction, we shall see in the brief Chapter 5 that this question reduces quite easily to the Kepler problem.

Chapter 6 investigates the central question of celestial mechanics, the n-body problem: How do n point masses move in $\mathbb{R}^3$ under mutual Newtonian attraction? We find some preserved quantities of this problem that allow us to make certain statements about the long-time behaviour of n-body systems, although we remain far from finding concrete solutions. In a section on central configurations I exhibit explicit solutions under additional geometric assumptions.

Chapter 7 deals with the special case $n = 3$. The centre-piece of that chapter is Lagrange's beautiful theorem on homographic (i.e. self-similar) solutions of the three-body problem. I also discuss the restricted three-body problem, where one of the three masses is negligibly small compared with the others.

In Chapter 8 we return to the Kepler problem, but from a more geometric point of view. This is really the geometric heart of the present text, where several types of geometric transformations (inversion, stereographic projection, polar reciprocation), spaces (hyperbolic space, projective plane) and differential geometric concepts (geodesics, curvature, conformal maps) are introduced. These techniques are used not only to give alternative proofs of Kepler's first law, but chiefly to give a unified view of all Kepler solutions, including the collision orbits (theorems of Moser, Osipov, and Belbruno).

Chapter 9 prepares the reader for the modern literature on the n-body problem by introducing the Hamiltonian formalism, starting from variational principles. In Chapter 10, the Hamiltonian formalism is applied to the Kepler problem. We determine the topology of the three-dimensional energy hypersurfaces in this problem, and I present a number of equivalent topological descriptions of these 3-manifolds. In particular, I use the quaternions to identify the special orthogonal group SO(3) as projective 3-space. Energy hypersurfaces with this topology also arise in the restricted three-body problem.

All chapters but one end with extensive historical notes and references.

Notational conventions

Vector quantities will be denoted in bold face; the euclidean length of a vector quantity is usually denoted by the corresponding symbol in italics. For example, $\mathbf{r}$ denotes the position vector of a particle in $\mathbb{R}^3$, and $r := |\mathbf{r}|$. The norm $|\,.\,|$ will always be the euclidean one. The standard (euclidean) inner product on $\mathbb{R}^3$ will be denoted by $\langle\,.\,,\,.\,\rangle$.

Time derivatives will be written with dots in the Newtonian fashion. For instance, if $t \mapsto \mathbf{r}(t)$ denotes the motion of a particle, its velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are given by

$$\mathbf{v} := \dot{\mathbf{r}} := \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \quad \mathbf{a} := \ddot{\mathbf{r}} := \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}.$$

The length $v := |\mathbf{v}|$ of the velocity vector is called the speed.

The natural numbers $\mathbb{N}$ are the positive integers; if 0 is to be included, I write $\mathbb{N}_0$. The rational, real and complex numbers are denoted by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. The positive reals are denoted by $\mathbb{R}^+$; the negative reals, by $\mathbb{R}^-$. We set $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$ and $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$. The notation $\mathbb{H}$ stands for hyperbolic space or Hamilton's quaternions, depending on the context. I use the standard notation C^k, $k \in \mathbb{N}$, for k times continuously differentiable functions or maps. By C^0 I simply mean continuous. Functions or maps of class C^∞ are also referred to as **smooth**.

Physical background

No prior knowledge of physics will be assumed apart from the following two Newtonian laws.

The second Newtonian law of motion: The acceleration $\mathbf{a}$ experienced by a body of mass m under the influence of a force $\mathbf{F}$ is given by

$$\mathbf{F} = m\mathbf{a}.$$

The universal law of gravitation: The force exerted by a body of mass m_2 at the point $\mathbf{r}_2 \in \mathbb{R}^3$ on a body of mass m_1 at the point $\mathbf{r}_1 \in \mathbb{R}^3$ equals

$$\mathbf{F} = \frac{Gm_1m_2}{r^2} \cdot \frac{\mathbf{r}}{r},$$

where $\mathbf{r} := \mathbf{r}_2 - \mathbf{r}_1$, and

$$G \approx 6.673 \cdot 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\ s}^2}$$

is the universal gravitational constant.

Mathematical background

I have tried to keep the mathematical prerequisites to a minimum, but the level of sophistication certainly increases as this text proceeds. A great number of the students taking my class at the University of Cologne were physics undergraduates in the second year of their studies. In their first year, they had followed my course on analysis and linear algebra, where they had seen, amongst other things, basic topological concepts, the notion of local and global diffeomorphisms, the inverse and the implicit function theorems, the classical matrix groups, elementary ordinary differential equations (the Picard–Lindelöf theorem on local existence and uniqueness, linear systems with constant coefficients), submanifolds, the transformation formula for higher-dimensional integrals, the integral theorems of Gauß and Stokes, and differential forms. In this text, submanifolds make a brief appearance in Chapter 7 and in the exercises to Chapter 8; the concept is essential for Section 9.2 and Chapter 10. Differential forms are used only in Section 9.2. Homeomorphisms (i.e. bijective maps that are continuous in either direction) and diffeomorphisms between submanifolds make a brief appearance in Section 8.3, and they become central only in Chapter 10. In that last chapter I also assume a certain familiarity with basic notions in point-set topology (Hausdorff property, compactness); the relevant material can be found in (Jänich, 2005) or (McCleary, 2006). In the context of an alternative proof of Kepler's first law, holomorphic maps appear in a couple of isolated places in the exercises to Chapter 8 and in Section 9.1.

As regards differential equations, throughout I use the following geometric interpretation. Let $\Omega \subset \mathbb{R}^d$ be an open subset and $\mathbf{X}$ a **vector field** on Ω, i.e. a function $\mathbf{X}\colon \Omega \to \mathbb{R}^d$. This gives rise to a first-order differential equation

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}).$$

Solutions of this differential equations are C^1-maps $\mathbf{x}\colon I \to \Omega$, defined on some interval $I \subset \mathbb{R}$, that satisfy this equation; that is,

$$\dot{\mathbf{x}}(t) = \mathbf{X}(\mathbf{x}(t)) \quad \text{for all } t \in I.$$

In geometric terms this means that $\mathbf{x}$ is an **integral curve** or **flow line** of $\mathbf{X}$, i.e. a curve whose velocity vector $\dot{\mathbf{x}}(t)$ at the point $\mathbf{x}(t)$ coincides with the vector $\mathbf{X}(\mathbf{x}(t))$ defined by the vector field $\mathbf{X}$ at that point.

The Picard–Lindelöf existence and uniqueness theorem (known to French readers as the Cauchy–Lipschitz theorem) says that if $\mathbf{X}$ is locally Lipschitz continuous, then for any $\mathbf{x}_0 \in \Omega$ the initial value problem

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0$$

has a solution defined on some small time interval $(-\delta, \delta)$, and two such solutions coincide on the time interval around 0 where both are defined. In all cases studied in this text, the vector field will actually be C^1 (or even smooth), so that local Lipschitz continuity is guaranteed by the mean value theorem.

Excellent texts on differential equations emphasising the geometric viewpoint are (Arnol'd, 1973) and (Bröcker, 1992). I can also recommend (Givental, 2001) and (Robinson, 1999). An eminently readable proof of the Picard–Lindelöf theorem is given in Appendix A of (Borrelli and Coleman, 2004).

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Cologne, November 2015 Hansjörg Geiges