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Christopher D. Sogge
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Background

The purpose of this chapter and the next is to present the background material that will be needed. The topics are standard and a more thorough treatment can be found in many excellent sources, such as Stein [2] and Stein and Weiss [1] for the first half and Hörmander [7, Vol. 1] for the second.

We start out by rapidly going over basic results from real analysis, including standard theorems concerning the Fourier transform in $\mathbb{R}^n$ and Calderón–Zygmund theory. We then apply this to prove the Hardy–Littlewood–Sobolev inequality. This theorem on fractional integration will be used throughout and we shall also present a simple argument showing how the n-dimensional theorem follows from the original one-dimensional inequality of Hardy and Littlewood. This type of argument will be used again and again. Finally, in the last two sections we give the definition of the wave front set of a distribution and compute the wave front sets of distributions which are given by oscillatory integrals. This will be our first encounter with the cotangent bundle and, as the monograph progresses, this will play an increasingly important role.

0.1 Fourier Transform

Given $f \in L^1(\mathbb{R}^n)$, we define its Fourier transform by setting

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\, dx. \tag{0.1.1}$$

Given $h \in \mathbb{R}^n$, let $(\tau_h f)(x) = f(x+h)$. Notice that $\tau_{-h} e^{-i\langle \cdot,\xi\rangle} = e^{i\langle h,\xi\rangle} e^{-i\langle \cdot,\xi\rangle}$ and so

$$(\tau_h f)^\wedge(\xi) = e^{i\langle h,\xi\rangle} \hat{f}(\xi). \tag{0.1.2}$$

In a moment, we shall see that we can invert (0.1.1) (for appropriate f) and that we have the formula

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{f}(\xi)\, d\xi. \tag{0.1.3}$$

Thus, the Fourier transform decomposes a function into a continuous sum of characters (eigenfunctions for translations).

Before turning to Fourier's inversion formula (0.1.3), let us record some elementary facts concerning the Fourier transform of L^1 functions.

Theorem 0.1.1

(1) $||\hat{f}||_\infty \le ||f||_1$.
(2) *If $f \in L^1$, then $\hat{f}$ is uniformly continuous.*

Theorem 0.1.2 (Riemann–Lebesgue) *If $f \in L^1(\mathbb{R}^n)$, then $\hat{f}(\xi) \to 0$ as $\xi \to \infty$, and hence, $\hat{f} \in C_0(\mathbb{R}^n)$.*

Theorem 0.1.1 follows directly from the definition (0.1.1). To prove Theorem 0.1.2, one first notices from an explicit calculation that the result holds when f is the characteristic function of a cube. From this one derives Theorem 0.1.2 via a limiting argument.

Even though $\hat{f}$ is in C_0, the integral (0.1.3) will not converge for general $f \in L^1$. However, for a dense subspace we shall see that the integral converges absolutely and that (0.1.3) holds.

Definition 0.1.3 The set of Schwartz-class functions, $\mathcal{S}(\mathbb{R}^n)$, consists of all $\phi \in C^\infty(\mathbb{R}^n)$ satisfying

$$\sup_x |x^\gamma \partial^\alpha \phi(x)| < \infty, \tag{0.1.4}$$

for all multi-indices α, γ.[1]

We give $\mathcal{S}$ the topology arising from the semi-norms (0.1.4). This makes $\mathcal{S}$ a Fréchet space. Notice that the set of all compactly supported C^∞ functions, $C_0^\infty(\mathbb{R}^n)$, is contained in $\mathcal{S}$.

Let $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. Then we have:

Theorem 0.1.4 *If $\phi \in \mathcal{S}$, then the Fourier transform of $D_j\phi$ is $\xi_j\hat{\phi}(\xi)$. Also, the Fourier transform of $x_j\phi$ is $-D_j\hat{\phi}$.*

[1] Here $\alpha = (\alpha_1,\dots,\alpha_n), \gamma = (\gamma_1,\dots,\gamma_n)$ while $x^\gamma = x_1^{\gamma_1}\cdots x_n^{\gamma_n}$ and $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$.

Proof To prove the second assertion we differentiate (0.1.1) to obtain

$$D_j\hat{\phi}(\xi) = \int e^{-i\langle x,\xi\rangle}(-x_j)\phi(x)\,dx,$$

since the integral converges uniformly. If we integrate by parts, we see that

$$\xi_j\hat{\phi}(\xi) = \int -D_j e^{-i\langle x,\xi\rangle}\cdot\phi(x)\,dx = \int e^{-i\langle x,\xi\rangle}D_j\phi(x)\,dx,$$

which is the first assertion. □

Notice that Theorem 0.1.4 implies the formula

$$\xi^\alpha D^\gamma\hat{\phi}(\xi) = \int e^{-i\langle x,\xi\rangle}D^\alpha((-x)^\gamma\phi(x))\,dx. \tag{0.1.5}$$

If we set $C = \int(1+|x|)^{-n-1}dx$, then this leads to the estimate

$$\sup_\xi|\xi^\gamma D^\alpha\hat{\phi}(\xi)| \le C\sup_x(1+|x|)^{n+1}|D^\gamma(x^\alpha\phi(x))|. \tag{0.1.6}$$

Inequality (0.1.6) of course implies that the Fourier transform maps $\mathcal{S}$ into itself. However, much more is true:

Theorem 0.1.5 *The Fourier transform $\phi\to\hat{\phi}$ is an isomorphism of $\mathcal{S}$ into $\mathcal{S}$ whose inverse is given by Fourier's inversion formula* (0.1.3).

The proof is based on a couple of lemmas. The first is the multiplication formula for the Fourier transform:

Lemma 0.1.6 *If $f,g\in L^1$ then*

$$\int_{\mathbb{R}^n}\hat{f}g\,dx = \int_{\mathbb{R}^n}f\hat{g}\,dx.$$

The next is a formula for the Fourier transform of Gaussians:

Lemma 0.1.7 $\int_{\mathbb{R}^n}e^{-i\langle x,\xi\rangle}e^{-\varepsilon|x|^2/2}dx = (2\pi/\varepsilon)^{n/2}e^{-|\xi|^2/2\varepsilon}$.

The first lemma is easy to prove. If we apply (0.1.1) and Fubini's theorem, we see that the left side equals

$$\int\left\{\int f(y)e^{-i\langle x,y\rangle}\,dy\right\}g(x)\,dx = \int\left\{\int e^{-i\langle x,y\rangle}g(x)\,dx\right\}f(y)\,dy$$
$$= \int\hat{g}f\,dy.$$

It is also clear that Lemma 0.1.7 must follow from the special case where $n = 1$. But

$$\int_{-\infty}^{\infty} e^{-t^2/2} e^{-it\tau}\, dt = e^{-\tau^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+i\tau)^2}\, dt$$
$$= e^{-\tau^2/2} \int_{-\infty}^{\infty} e^{-t^2/2} dt$$
$$= \sqrt{2\pi}\, e^{-\tau^2/2}.$$

In the second step we have used Cauchy's theorem. If we make the change of variables $\varepsilon^{1/2}s = t$ in the last integral, we get the desired result.

Proof of Theorem 0.1.5 We must prove that when $\phi \in \mathcal{S}$,

$$\phi(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} \hat{\phi}(\xi)\, d\xi.$$

By the dominated convergence theorem, the right side equals

$$\lim_{\varepsilon \to 0} (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} \hat{\phi}(\xi) e^{-\varepsilon|\xi|^2/2} d\xi.$$

If we recall (0.1.2), then we see that this equals

$$\lim_{\varepsilon \to 0} (2\pi\varepsilon)^{-n/2} \int \phi(x+y) e^{-|y|^2/2\varepsilon} dy.$$

Finally, since $(2\pi)^{-n/2} \int e^{-|y|^2/2} dy = 1$, it is easy to check that the last limit is $\phi(x)$. □

If for $f, g \in L^1$ we define convolution by

$$(f * g)(x) = \int f(x-y) g(y)\, dy,$$

then another fundamental result is:

Theorem 0.1.8 *If $\phi, \psi \in \mathcal{S}$ then*

$$(2\pi)^n \int \phi \overline{\psi}\, dx = \int \hat{\phi} \overline{\hat{\psi}}\, d\xi, \tag{0.1.7}$$

$$(\phi * \psi)^\wedge(\xi) = \hat{\psi}(\xi) \hat{\psi}(\xi), \tag{0.1.8}$$

$$(\phi\psi)^\wedge(\xi) = (2\pi)^{-n} (\hat{\phi} * \hat{\psi})(\xi). \tag{0.1.9}$$

To prove (0.1.7), set $\chi = (2\pi)^{-n}\overline{\hat{\psi}}$. Then the Fourier inversion formula implies that $\hat{\chi} = \overline{\psi}$. Consequently, (0.1.7) follows from Lemma 0.1.6. We leave the other two formulas as exercises.

We shall now discuss the Fourier transform of more general functions. First, we make a definition.

Definition 0.1.9 The dual space of $\mathcal{S}$ is $\mathcal{S}'$. We call $\mathcal{S}'$ the space of tempered distributions.

Definition 0.1.10 If $u \in \mathcal{S}'$, we define its Fourier transform $\hat{u} \in \mathcal{S}'$ by setting, for all $\phi \in \mathcal{S}$,

$$\hat{u}(\phi) = u(\hat{\phi}). \tag{0.1.10}$$

Notice how Lemma 0.1.6 says that when $u \in L^1$, Definition 0.1.10 agrees with our previous definition of $\hat{u}$. Using Fourier's inversion formula for $\mathcal{S}$, one can check that $u \to \hat{u}$ is an isomorphism of $\mathcal{S}'$. If $u \in L^1$ and $\hat{u} \in L^1$, we conclude that the inversion formula (0.1.3) must hold for almost all x.

Theorem 0.1.11 *If $u \in L^2$ then $\hat{u} \in L^2$ and*

$$\|\hat{u}\|_2^2 = (2\pi)^n \|u\|_2^2 \qquad \text{(Plancherel's theorem).} \tag{0.1.11}$$

Furthermore, Parseval's formula holds whenever $\phi, \psi \in L^2$:

$$\int \phi \overline{\psi}\, dx = (2\pi)^{-n} \int \hat{\phi} \overline{\hat{\psi}}\, d\xi. \tag{0.1.12}$$

Proof Choose $u_j \in \mathcal{S}$ satisfying $u_j \to u$ in L^2. Then, by (0.1.7),

$$\|\hat{u}_j - \hat{u}_k\|_2^2 = (2\pi)^n \|u_j - u_k\|_2^2 \to 0.$$

Thus, $\hat{u}_j$ converges to a function v in L^2. But the continuity of the Fourier transform in $\mathcal{S}'$ forces $v = \hat{u}$. This gives (0.1.11), since (0.1.11) is valid for each u_j. Since we have just shown that the Fourier transform is continuous on L^2, (0.1.12) follows from the fact that we have already seen that it holds when ϕ and ψ belong to the dense subspace $\mathcal{S}$. □

Since, for $1 \le p \le 2, f \in L^p$ can be written as $f = f_1 + f_2$ with $f_1 \in L^1, f_2 \in L^2$, it follows from Theorem 0.1.1 and Theorem 0.1.11 that $\hat{f} \in L^2_{\text{loc}}$. A much better result is:

Theorem 0.1.12 (Hausdorff–Young) *Let $1 \le p \le 2$ and define p' by $1/p + 1/p' = 1$. Then, if $f \in L^p$ it follows that $\hat{f} \in L^{p'}$ and*

$$\|\hat{f}\|_{p'} \le (2\pi)^{n/p'} \|f\|_p.$$

Since we have already seen that this result holds for $p = 1$ and $p = 2$, this follows from:

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Theorem 0.1.13 (M. Riesz interpolation theorem) *Let T be a linear map from $L^{p_0}\cap L^{p_1}$ to $L^{q_0}\cap L^{q_1}$ satisfying*

$$\|Tf\|_{q_j} \le M_j\|f\|_{p_j}, \qquad j=0,1, \tag{0.1.13}$$

with $1\le p_j,q_j\le\infty$. Then, if for $0<t<1, 1/p_t=(1-t)/p_0+t/p_1, 1/q_t=(1-t)/q_0+t/q_1$,

$$\|Tf\|_{q_t} \le (M_0)^{1-t}(M_1)^t\|f\|_{p_t}, \qquad f\in L^{p_0}\cap L^{p_1}. \tag{0.1.14}$$

Proof If $p_t=\infty$ the result follows from Hölder's inequality since then $p_0=p_1=\infty$. So we shall assume that $p_t<\infty$.

By polarization it then suffices to show that

$$\left|\int Tfg\,dx\right| \le M_0^{1-t}M_1^t\|f\|_{p_t}\|g\|_{q_t'}, \tag{0.1.15}$$

when f and g vanish outside of a set of finite measure and take on a finite number of values, that is, $f=\sum_{j=1}^m a_j\chi_{E_j}, g=\sum_{k=1}^N b_k\chi_{F_k}$, with $E_j\cap E_{j'}=\emptyset$ and $F_k\cap F_{k'}=\emptyset$ if $j\ne j'$ and $k\ne k'$ and $|E_j|,|F_k|<\infty$ for all j and k. We may also assume $\|f\|_{p_t},\|g\|_{q_t'}\ne 0$ and so, if we divide both sides by the norms, it suffices to prove (0.1.15) when $\|f\|_{p_t}=\|g\|_{q_t'}=1$.

Next, if $a_j=e^{i\theta_j}|a_j|$ and $b_k=e^{i\psi_k}|b_k|$, then, assuming $q_t>1$, we set

$$f_z=\sum_{j=1}^m |a_j|^{\alpha(z)/\alpha(t)}e^{i\theta_j}\chi_{E_j},$$

$$g_z=\sum_{k=1}^N |b_k|^{(1-\beta(z))/(1-\beta(t))}e^{i\psi_k}\chi_{F_k},$$

where $\alpha(z)=(1-z)/p_0+z/p_1$ and $\beta(z)=(1-z)/q_0+z/q_1$. If $q_t=1$ then we modify the definition by taking $g_z\equiv g$. It then follows that $F(z)=\int Tf_zg_z\,dx$ is entire and bounded in the strip $0\le \mathrm{Re}(z)\le 1$. Also, $F(t)$ equals the left side of (0.1.15). Consequently, by the three-lines lemma,[2] we would be done if we could prove

$$|F(z)|\le M_0, \qquad \mathrm{Re}(z)=0,$$
$$|F(z)|\le M_1, \qquad \mathrm{Re}(z)=1.$$

To prove the first inequality, notice that for $y\in\mathbb{R}$, $\alpha(iy)=1/p_0+iy(1/p_1-1/p_0)$. Consequently,

$$|f_{iy}|^{p_0}=|e^{i\arg f}\cdot|f|^{iy(1/p_1-1/p_0)}\cdot|f|^{p_t/p_0}|^{p_0}=|f|^{p_t}.$$

[2] See, for example, Stein and Weiss [1, p. 180].

Similar considerations show that $|g_{iy}|^{q_0'} = |g|^{q_t'}$. Applying Hölder's inequality and (0.1.13) gives

$$\begin{aligned}|F(iy)| &\le \|Tf_{iy}\|_{q_0}\|g_{iy}\|_{q_0'}\\ &\le M_0\|f_{iy}\|_{p_0}\|g_{iy}\|_{q_0'} = M_0\|f\|_{p_t}^{p_t/p_0}\|g\|_{q_t'}^{q_t'/q_0'} = M_0.\end{aligned}$$

Since a similar argument gives the other inequality, we are done. □

Later on it will be important to know how the Fourier transform behaves under linear changes of variables. If $T:\mathbb{R}^n\to\mathbb{R}^n$ is a linear bijection and $u\in\mathcal{S}'(\mathbb{R}^n)\cap C(\mathbb{R}^n)$, we can define its pullback under T by

$$T^*u = u\circ T.$$

Note that a change of variables gives

$$\begin{aligned}(T^*u)(\phi) &= \int u(Tx)\phi(x)\,dx\\ &= \int u(y)\,|\det T^{-1}|\,\phi(T^{-1}y)\,dy = u(|\det T^{-1}|\,\phi(T^{-1}\cdot)),\end{aligned}$$

so, for general $u\in\mathcal{S}'(\mathbb{R}^n)$, we define the pullback using the left and right sides of this equality.

Theorem 0.1.14 *With the above notation*

$$(T^*u)^\wedge = |\det T|^{-1}({}^tT^{-1})^*\hat{u}. \tag{0.1.16}$$

We leave the proof as an exercise. As a consequence we have:

Corollary 0.1.15 *If $u\in\mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree σ, then $\hat{u}$ is homogeneous of degree $-n-\sigma$.*

Proof u being homogeneous of degree σ means that if $M_tx = tx$, then $M_t^*u = t^\sigma u$. So, by Theorem 0.1.14, $t^\sigma\hat{u} = (M_t^*u)^\wedge = t^{-n}M_{1/t}^*\hat{u}$. If we replace t by $1/t$ this means that $M_t^*\hat{u} = t^{-n-\sigma}\hat{u}$. □

Remark Notice that if Re $\sigma < -n$, then $\hat{u}$ is continuous. Using this and Theorem 0.1.4, the reader can check that if u is homogeneous and in $C^\infty(\mathbb{R}^n\backslash 0)$, then so is $\hat{u}$.

Let us conclude this section by presenting the Poisson summation formula. If g is a function on $\mathbb{R}^n$, then we shall say that g is periodic (with period 2π) if $g(x+2\pi m) = g(x)$ for all $m\in\mathbb{Z}^n$. Given, say, $\phi\in\mathcal{S}(\mathbb{R}^n)$, there are two ways that one can construct a periodic function out of ϕ. First, one could set

$g = \sum_{m\in\mathbb{Z}^n} \phi(x+2\pi m)$; or one could take $g = (2\pi)^{-n} \sum_{m\in\mathbb{Z}^n} \hat{\phi}(m)e^{i\langle x,m\rangle}$. Notice that both series converge uniformly to a periodic C^∞ function in view of the rapid decrease of ϕ and $\hat{\phi}$. The Poisson summation formula says that the two periodic extensions are the same.

Theorem 0.1.16 *If $\phi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$\sum_{m\in\mathbb{Z}^n} \phi(x+2\pi m) = (2\pi)^{-n} \sum_{m\in\mathbb{Z}^n} \hat{\phi}(m)e^{i\langle x,m\rangle}.$$

In particular, we have the Poisson summation formula:

$$\sum_{m\in\mathbb{Z}^n} \phi(2\pi m) = (2\pi)^{-n} \sum_{m\in\mathbb{Z}^n} \hat{\phi}(m). \tag{0.1.17}$$

To prove this result, let $\mathbb{T}^n = 2\pi(\mathbb{R}^n/\mathbb{Z}^n)$. Then, if we set $Q = [-\pi,\pi]^n$, it is clear that the series $\sum_{m\in\mathbb{Z}^n} \phi(x+2\pi m) = g$ converges uniformly in the $L^1(Q)$ norm. Thus, for $k \in \mathbb{Z}^n$, its Fourier coefficients are given by

$$\begin{aligned} g_k &= \int_Q e^{-i\langle x,k\rangle} g(x)\,dx = \int_Q e^{-i\langle x,k\rangle} \sum_{m\in\mathbb{Z}^n} \phi(x+2\pi m)\,dx \\ &= \sum_{m\in\mathbb{Z}^n} \int_Q e^{-i\langle x,k\rangle} \phi(x+2\pi m)\,dx = \sum_{m\in\mathbb{Z}^n} \int_{Q+2\pi m} e^{-i\langle x,k\rangle} \phi(x)\,dx \\ &= \int_{\mathbb{R}^n} e^{-i\langle x,k\rangle} \phi(x)\,dx = \hat{\phi}(k). \end{aligned}$$

On the other hand, if we set $(2\pi)^{-n} \sum_{m\in\mathbb{Z}^n} \hat{\phi}(m)e^{i\langle x,m\rangle} = \tilde{g}$, then the series also converges uniformly in $L^1(Q)$. Its Fourier coefficients are

$$\begin{aligned} \tilde{g}_k &= \int_Q e^{-i\langle x,k\rangle} (2\pi)^{-n} \sum_{m\in\mathbb{Z}^n} \hat{\phi}(m)e^{i\langle x,m\rangle}\,dx \\ &= (2\pi)^{-n} \sum_{m\in\mathbb{R}^n} \hat{\phi}(m) \int_Q e^{i\langle x,m-k\rangle}\,dx \\ &= (2\pi)^{-n} \sum_{m\in\mathbb{Z}^n} \hat{\phi}(m)\cdot(2\pi)^n \delta_{k,m} = \hat{\phi}(k). \end{aligned}$$

Thus, since g and $\tilde{g}$ have the same Fourier coefficients, we would be done if we could prove:

Lemma 0.1.17 *If μ is a Borel measure on $\mathbb{T}^n$ satisfying $\int_{\mathbb{T}^n} e^{-i\langle x,k\rangle}\,d\mu(x) = 0$ for all $k \in \mathbb{Z}^n$, then $\mu = 0$.*

To prove this, we first notice that, by the Stone–Weierstrass theorem, trigonometric polynomials are dense in $C(\mathbb{T}^n)$, since they form an algebra that separates points and is closed under complex conjugation. Our hypothesis implies that $\int_{\mathbb{T}^n} P(x)\,d\mu(x) = 0$ whenever P is a trigonometric polynomial. The approximation property then implies that $\int_{\mathbb{T}^n} f(x)\,d\mu(x) = 0$ for any $f \in C(\mathbb{T}^n)$. By the Riesz representation theorem $\mu = 0$.

Using the Poisson summation theorem one can recover basic facts about Fourier series. For instance:

Theorem 0.1.18 *If $g \in L^2(\mathbb{T}^n)$ then $(2\pi)^{-n}\sum_{k\in\mathbb{Z}^n} g_k e^{i\langle x,k\rangle}$ converges to g in the L^2 norm and we have Parseval's formula*

$$\int_{\mathbb{T}^n} |g|^2\,dx = (2\pi)^{-n} \sum_{k\in\mathbb{Z}^n} |g_k|^2.$$

Conversely, if $\sum |g_k|^2 < \infty$, then $(2\pi)^{-n}\sum_{k\in\mathbb{Z}^n} g_k e^{i\langle x,k\rangle}$ converges to an L^2 function with Fourier coefficients g_k.

Proof If $g \in C^\infty(\mathbb{T}^n)$ then the Poisson summation formula implies that, if we identify $\mathbb{T}^n$ and Q as above, then for $x \in Q$,

$$(2\pi)^{-n} \sum_{k\in\mathbb{Z}^n} g_k e^{i\langle x,k\rangle} = \sum_{k\in\mathbb{Z}^n} g(x + 2\pi k).$$

Hence, if $g \in C^\infty(\mathbb{T}^n)$, its Fourier series converges to g uniformly, since one can check by integration by parts that $g_k = O(|k|^{-N})$ for any N. Consequently,

$$\int_{\mathbb{T}^n} |g|^2\,dx = (2\pi)^{-2n} \int_{\mathbb{T}^n} \sum g_k \overline{g_{k'}} e^{i\langle x,k-k'\rangle}\,dx = (2\pi)^{-n} \sum |g_k|^2.$$

Thus, the map sending $g \in L^2(\mathbb{T}^n)$ to its Fourier coefficients $g_k \in \ell^2(\mathbb{Z}^n)$ is an isometry. It is also unitary since the range contains the dense subspace $\ell^1(\mathbb{Z}^n)$. $\square$

0.2 Basic Real Variable Theory

In this section we shall study two basic topics in real variable theory: the boundedness of the Hardy–Littlewood maximal function and the boundedness of certain Fourier-multiplier operators. Since the Hardy–Littlewood maximal theorem is simpler and since a step in its proof will be used in the proof of the multiplier theorem, we shall start with it.

If ω_n denotes the volume of the unit ball B in $\mathbb{R}^n$, then, given $f \in L^1_{\text{loc}}$, we define the Hardy–Littlewood maximal function associated to f by

$$\mathcal{M}f(x) = \sup_{t>0} \int_B |f(x-ty)|\,\frac{dy}{\omega_n}. \tag{0.2.1}$$

If $B(x,t)$ denotes the ball of radius t centered at x then of course

$$M_t f(x) = \int_B f(x-ty)\,\frac{dy}{\omega_n} = \frac{1}{|B(x,t)|}\int_{B(x,t)} f(y)\,dy, \tag{0.2.2}$$

so in (0.2.1) we are taking the supremum of the mean values of $|f|$ over all balls centered at x. We have used the notation in (0.2.1) to be consistent with some generalizations to follow.

Theorem 0.2.1 (Hardy–Littlewood maximal theorem) *If $1 < p \le \infty$ then*

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)}. \tag{0.2.3}$$

Furthermore, $\mathcal{M}$ is not bounded on L^1; however,

$$|\{x : \mathcal{M}f(x) > \alpha\}| \le C\alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)}. \tag{0.2.4}$$

As a consequence, we obtain Lebesgue's differentiation theorem:

Corollary 0.2.2 *If $f \in L^1_{\text{loc}}$, then for almost every x*

$$\lim_{t\to 0}\int_B f(x-ty)\,\frac{dy}{\omega_n} = f(x). \tag{0.2.5}$$

Before proving the Hardy–Littlewood maximal theorem, let us give the simple argument showing how it implies the corollary. First, it is clear that, in order to prove (0.2.5), it suffices to consider only compactly supported f. Hence, we may assume $f \in L^1(\mathbb{R}^n)$ and that f is real valued.

Next, let us set

$$f^*(x) = |\limsup_{t\to 0} M_t f(x) - \liminf_{t\to 0} M_t f(x)|.$$

For $g \in L^1, g^*(x) \le 2\mathcal{M}g(x)$. Consequently, (0.2.4) gives

$$|\{x : g^*(x) > \alpha\}| \le 2C\alpha^{-1}\|g\|_{L^1}.$$

To finish matters, we use the fact that, given $\varepsilon > 0$, any $f \in L^1$ can be written as $f = g + h$ with $h \in C(\mathbb{R}^n)$ and $\|g\|_{L^1} < \varepsilon$. Clearly $h^* \equiv 0$, and so

$$|\{x : f^*(x) > \alpha\}| = |\{x : g^*(x) > \alpha\}| \le 2C\alpha^{-1}\varepsilon.$$

Since ε is arbitrary, we conclude that $f^* = 0$ almost everywhere, which of course gives (0.2.5).

Turning to the theorem, we leave it as an exercise for the reader that if $f = \chi_B$ then $\|\mathcal{M}f\|_{L^1} = +\infty$. On the other hand, to prove the substitute, (0.2.4), we shall require: