

## NONLINEAR AND STOCHASTIC CLIMATE DYNAMICS

It is now widely recognized that the climate system is governed by nonlinear, multi-scale processes, whereby memory effects and stochastic forcing by fast processes, such as weather and convective systems, can induce regime behavior. Motivated by present difficulties in understanding the climate system and to aid the improvement of numerical weather and climate models, this book gathers contributions from mathematics, physics, and climate science to highlight the latest developments and current research questions in nonlinear and stochastic climate dynamics. Leading researchers discuss some of the most challenging and exciting areas of research in the mathematical geosciences, such as the theory of tipping points and of extreme events including spatial extremes, climate networks, data assimilation, and dynamical systems. This book provides graduate students and researchers with a broad overview of the physical climate system and introduces powerful data analysis and modeling methods for climate scientists and applied mathematicians.

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# NONLINEAR AND STOCHASTIC CLIMATE DYNAMICS

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CAMBRIDGE  
UNIVERSITY PRESS

## CAMBRIDGE UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom  
 One Liberty Plaza, 20th Floor, New York, NY 10006, USA  
 477 Williamstown Road, Port Melbourne, VIC 3207, Australia  
 4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India  
 79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.  
 It furthers the University’s mission by disseminating knowledge in the pursuit of  
 education, learning, and research at the highest international levels of excellence.

www.cambridge.org  
 Information on this title: www.cambridge.org/9781107118140  
 DOI: 10.1017/9781316339251

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First published 2017

*A catalogue record for this publication is available from the British Library.*

*Library of Congress Cataloging-in-Publication Data*

Names: Franzke, Christian L. E., editor. | O’Kane, Terence J., editor.  
 Title: Nonlinear and stochastic climate dynamics / edited by Christian L.E.  
 Franzke, University of Hamburg, Germany, and Terence J. O’Kane,  
 Marine and Atmospheric Research, CSIRO, Australia.  
 Description: New York, NY : Cambridge University Press, [2017] |  
 Includes index.  
 Identifiers: LCCN 2016045370 | ISBN 9781107118140  
 Subjects: LCSH: Statistical weather forecasting. | Weather forecasting. |  
 Global warming. | Climatic changes. | Atmospheric  
 circulation – Mathematical models.  
 Classification: LCC QC996.5 .N66 2017 | DDC 551.601/51922–dc23  
 LC record available at <https://lccn.loc.gov/2016045370>

ISBN 978-1-107-11814-0 Hardback

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$$D_1 = D_{KY} = K + \frac{\sum_{k=1}^K \lambda_k}{|\lambda_{K+1}|}, \tag{14.9}$$

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## Preface

Unquestionably the climate system is a highly nonlinear and complex system characterized by the communication of coherent information across a multitude of scales and exhibiting on occasion abrupt transitions between periods of relative stability. Given the importance of the potential societal impacts of anthropogenic influences, knowledge of both the past and future evolution of the climate system is of utmost importance. As a consequence there has been increased demand for predictions on the timescales from seasons to decades. To underpin predictive models beyond the timescales of synoptic weather forecasts, there is a huge need for applied mathematical, statistical, network, and complexity approaches to further our understanding of where predictive skill resides in the climate system.

The nonlinear, stochastic, and multiscale nature of the problem makes development of a foundational theoretical basis for climate variability and predictability a distinct challenge. Mathematicians, geoscientists, and fluid dynamicists are by nature attracted to the “hard problems” and this is reflected in the growing richness of approaches and increasing numbers of interdisciplinary mathematical and geoscience researchers attending the Nonlinear Processes sessions of the two big geosciences unions, the European Geoscience Union and the American Geophysical Union and dedicated conferences like SIAMs Mathematics of Planet Earth.

Traditional climate science has understood many important aspects of the climate system by linearizing the equations of motions and by making Markovian and Gaussian assumptions in combination with a deterministic view of the climate system. However, there is increasing observational evidence that the climate system is highly nonlinear, non-stationary, and non-Gaussian and that in many areas memory (non-Markovian) effects are important. These effects render our ability to make skillful predictions based on linear methods highly problematic. It has been shown that stochastic parameterizations can significantly improve ensemble prediction and data assimilation schemes. More generally, stochastic methods are particularly well suited for the analysis of high dimensional, non-stationary multiscale systems of which the atmosphere and ocean are two paradigmatic examples. Furthermore, stochastic methods have been shown to become crucial to understanding the impact of small-scale growing disturbances on the large-scale quasi-stationary structures manifest in the climate.

While physics, mathematics, and statistics contribute more and more to a nonlinear and stochastic view of the climate system, these communities still do not have a very close relationship with the climate science community, in part due to a lack of dedicated journals and books at the intersections of these communities. This book aims to link these communities and to give graduate students and interested scientists an accessible entry into nonlinear and stochastic climate dynamics by highlighting the latest developments and current research questions.

The contributions in this book cover the latest developments in this exciting research area of climate dynamics. The first two chapters deal with the particularly nonlinear (nonsecular) trends evident in the paleoclimate record. We begin with the chapter (1) by Crucifix et al. discussing how the nonlinearity of the climate system hampers predictions of the future climate, and in particular, the onset of the next ice age. The slow dynamics of ice ages feature the nonlinear interaction of ice sheets, the deep ocean, and carbon cycle dynamics. One important conclusion of this contribution is the usefulness of stochastic parameterizations in climate modeling. The second chapter by Ditlevsen discusses predictability in the climate system using the ice core record. The Greenland ice cores show a number of abrupt changes which are nonlinear responses inherent in the climate system and which can be interpreted as being tipping points. Ditlevsen shows that the predictability of tipping points depends on their dynamical origin and whether they are bifurcation (deterministic chaos) or noise induced. While the underlying dynamics of the transitions might not be exactly known, early warming systems for tipping points might still have some predictive skill.

The so-called teleconnection patterns or persistent regime states of the large-scale atmosphere and ocean are perhaps the most well recognized and most important area of nonlinear climate dynamics. In chapter 3 Feldstein and Franzke review the history and dynamics of atmospheric teleconnection patterns and regime states. For instance, the importance of the North Atlantic Oscillation teleconnection pattern and its surface impacts has been known since the time of the Vikings in the 12th century. Additionally, surface impacts of teleconnection patterns are associated with extremes of heat and cold, and have application for statistical downscaling, long-range predictions and paleoclimate reconstructions. While there is currently no comprehensive theory of teleconnection patterns, Feldstein and Franzke review the current understanding of the mechanisms driving them. Straus et al. (Chapter 4) extend the focus on atmospheric regimes considering their link to weather and the large-scale circulation including an examination of cluster methods used to identify regime states. This close link is particularly important for long-range predictions and impact studies.

While it is well recognized that regime transitions in climate systems are often initiated by an instability in the flow, predictability and the mechanisms underlying regime transitions are in general not well understood. Nadiga and O’Kane (Chapter 5) present the results of a regime and predictability study in an idealized barotropic vorticity model that admits low-frequency regime transitions between zonal and dipolar states. They consider perturbations that are embedded onto the system’s chaotic attractor under the full

nonlinear dynamics as nonlinear generalizations of the leading (backward) Lyapunov vector and where transitions between regimes are initiated by weak stochastic forcing of the large-scale modes.

Donner et al. (Chapter 6) take a particular nonlinear view on preferred atmospheric circulation patterns by applying network theory. In their contribution they point out similarities between network approaches and more traditional methods, which are mainly based on assumptions of linearity and stationarity, like Empirical Orthogonal Functions (EOFs). While the previous approaches are mainly aiming for preferred or recurring patterns, Horenko et al. (Chapter 7) discuss methods for identifying persistent patterns which likely are more predictable. Their non-parametric variational approach is especially suited for multiscale data and does not make assumptions about the underlying dynamics of the regime states and their evolution, i.e. stationarity.

A recent and exciting development is the use of stochastic methods in climate research. Gottwald et al. (Chapter 8) present the fundamentals of a stochastic climate theory. They show that the Mori-Zwanzig formalism, well-known from statistical physics, predicts the presence of memory effects in reduced order models. This implies that subgrid scale parameterizations for numerical weather and climate prediction models should include time-history effects. The current generation of operational weather and climate prediction models rarely consider such effects in parameterizations of subgrid processes. Gottwald et al. also review current methods to empirically estimate the coefficients for stochastic reduced order models. Frederiksen et al. (Chapter 9) review the development of a family of systematic parameter free approaches to subgrid scale parameterizations based on stochastic modeling and closure-based representations of the effects of subgrid turbulence in large eddy simulations of turbulent fluids including memory and non-Gaussian effects. Harlim (Chapter 10) presents the latest developments in data assimilation demonstrating that stochastic parameterizations are a particularly promising Ansatz for mitigating the effects of model error in data assimilation.

Bunde and Ludescher (Chapter 11) introduce the concept of long-memory in their contribution. Long-memory processes are those whose autocorrelation function decays according to a power-law. That memory effects are important in the climate system was already discussed in the chapter by Gottwald et al. (Chapter 8). The long-memory characteristic implies that perturbations can be long-term persistent. This property hinders the detection of significant climate trends. In particular, Bunde and Ludescher show that the first-order autoregressive process currently popular in trend testing are inadequate and that long-memory methods have to be used. The authors also show that long-memory also leads to the clustering of extreme events. While Bunde and Ludescher focus on Gaussian processes, Watkins (Chapter 12) surveys methods which simultaneously are long-memory and have heavy tails, i.e. have severely non-Gaussian distributions.

Ribatet (Chapter 13) presents the latest developments in the study of spatial extremes and the application of max-stable processes. Bodai (Chapter 14) uses a dynamical systems approach to extremes. In this approach the attractor dimension is related to the parameters of the classical extreme value distributions, well-known from statistical extreme value

theory. He discusses the link between extreme value statistics and the geometrical properties of the attractor in both high-dimensional systems and in very low-dimensional settings where the fractality of the attractor prevents the system from having an extreme value law.

In summary, this book provides, perhaps for the first time, an accessible overview on nonlinear and stochastic climate dynamics at the leading edge of research. We hope to contribute with this book to a fruitful interaction between the mathematical, physical, and climate science communities.

CF acknowledges generous support by the German Research Foundation (DFG) through the cluster of excellence CliSAP (EXC177) and the SFB/TRR181 “Energy transfers in atmosphere and ocean” throughout the preparation of this book.