

Contents

<i>List of Illustrations</i>	<i>page xi</i>
<i>Preface</i>	<i>xiii</i>
1 Hyperbolic space and its isometries	1
1.1 Möbius transformations	1
1.2 Hyperbolic geometry	6
1.2.1 The hyperbolic plane	8
1.2.2 Hyperbolic space	8
1.3 The circle or sphere at infinity	12
1.4 Gaussian curvature	16
1.5 Further properties of Möbius transformations	19
1.5.1 Commutativity	19
1.5.2 Isometric circles and planes	20
1.5.3 Trace identities	23
1.6 Exercises and explorations	24
2 Discrete groups	53
2.1 Convergence of Möbius transformations	53
2.1.1 Some group terminology	55
2.2 Discreteness	55
2.3 Elementary discrete groups	59
2.4 Kleinian groups	62
2.4.1 The limit set $\Lambda(G)$	62
2.4.2 The ordinary (regular, discontinuity) set $\Omega(G)$	64
2.5 Quotient manifolds and orbifolds	66
2.5.1 Covering surfaces and manifolds	67
2.5.2 Orbifolds	70
2.5.3 The conformal boundary	73
2.6 Two fundamental algebraic theorems	73
2.7 Introduction to Riemann surfaces and their uniformization	75
2.8 Fuchsian and Schottky groups	80
2.8.1 Handlebodies	82
2.9 Riemannian metrics and quasiconformal mappings	84

viii	<i>Contents</i>	
	2.10	Teichmüller spaces of Riemann surfaces 87
	2.10.1	Teichmüller mappings 89
	2.11	The mapping class group $\mathcal{MCG}(R)$ 91
	2.11.1	Dehn twists 91
	2.11.2	The action of $\mathcal{MCG}(R)$ on R and $\mathcal{T}eich(R)$ 91
	2.11.3	The complex structure of $\mathcal{T}eich(R)$ 94
	2.12	Exercises and explorations 94
	2.12.1	Summary of group properties 114
3	Properties of hyperbolic manifolds	122
	3.1	The Ahlfors Finiteness Theorem 122
	3.2	Tubes and horoballs 123
	3.3	Universal properties in hyperbolic 3-manifolds and orbifolds 126
	3.4	The thick/thin decomposition of a manifold 134
	3.5	Fundamental polyhedra 135
	3.5.1	The Ford fundamental region and polyhedron 138
	3.5.2	Poincaré's Theorem 142
	3.5.3	The Cayley graph dual to tessellation 143
	3.6	Geometric finiteness 144
	3.6.1	Finite volume 149
	3.7	Three-manifold surgery 149
	3.7.1	Compressible and incompressible boundary 151
	3.7.2	Extensions $\partial\mathcal{M} \rightarrow \mathcal{M}$ 153
	3.8	Quasifuchsian groups 155
	3.8.1	Simultaneous uniformization 157
	3.9	Geodesic and measured geodesic laminations 158
	3.9.1	Geodesic laminations 158
	3.9.2	Measured geodesic laminations 161
	3.9.3	Geometric intersection numbers 162
	3.9.4	Length of measured laminations 164
	3.10	The convex hull of the limit set 167
	3.10.1	The bending measure 169
	3.10.2	Pleated surfaces 170
	3.11	The convex core 175
	3.11.1	Length estimates for the convex core boundary 177
	3.11.2	Bending measures on convex core boundary 178
	3.12	The compact and relative compact core 180
	3.13	Rigidity of hyperbolic 3-manifolds 182
	3.14	Exercises and explorations 187
4	Algebraic and geometric convergence	219
	4.1	Algebraic convergence 219
	4.2	Geometric convergence 225
	4.3	Polyhedral convergence 226

<i>Contents</i>		ix
4.4	The geometric limit	229
4.5	Sequences of limit sets and regions of discontinuity	232
4.5.1	Hausdorff and Carathéodory convergence	232
4.5.2	Convergence of groups and regular sets	234
4.6	New parabolics	237
4.7	Acyindrical manifolds	239
4.8	Dehn filling and Dehn surgery	241
4.9	The prototypical example	242
4.10	Manifolds of finite volume	245
4.10.1	The Dehn Surgery Theorem	246
4.10.2	Sequences of volumes	249
4.10.3	Well ordering of volumes	250
4.10.4	Minimum volumes	251
4.11	Exercises and explorations	253
5	Deformation spaces and the ends of manifolds	276
5.1	The representation variety	276
5.1.1	The discreteness locus	279
5.1.2	The quasiconformal deformation space $\mathfrak{T}(G)$	280
5.2	Homotopy equivalence	281
5.2.1	Components of the discreteness locus	284
5.3	The quasiconformal deformation space boundary	287
5.3.1	Bumping and self-bumping	289
5.4	The three conjectures for geometrically infinite manifolds	289
5.5	Ends of hyperbolic manifolds	290
5.6	Tame manifolds	291
5.7	The Ending Lamination Theorem	296
5.8	The Double Limit Theorem	303
5.9	The Density Theorem	304
5.10	Bers slices	305
5.11	The quasifuchsian space boundary	309
5.11.1	The Bers (analytic) boundary	310
5.11.2	The Thurston (geometric) boundary	314
5.12	Examples of geometric limits at the Bers boundary	317
5.13	Classification of the geometric limits	323
5.14	Cannon-Thurston mappings	326
5.14.1	The Cannon-Thurston Theorem	326
5.14.2	Cannon-Thurston mappings and local connectivity	329
5.15	Exercises and explorations	332
6	Hyperbolization	371
6.1	Hyperbolic manifolds that fiber over a circle	371
6.1.1	Automorphisms of surfaces	371
6.1.2	Pseudo-Anosov mappings	372

6.1.3	The space of hyperbolic metrics	374
6.1.4	Fibering	374
6.2	Hyperbolic gluing boundary components	375
6.2.1	Skinning a bordered manifold	375
6.2.2	Totally geodesic boundary	376
6.2.3	Gluing boundary components	378
6.2.4	The Bounded Image Theorem	381
6.3	Hyperbolization of 3-manifolds	382
6.3.1	Review of definitions in 3-manifold topology	382
6.3.2	Hyperbolization	385
6.4	The three big conjectures, now theorems, for closed manifolds	386
6.4.1	Surface subgroups of $\pi_1(\mathcal{M}(G)) = G$	387
6.4.2	Remarks on the proof of VHT and VFT: Cubulation	390
6.4.3	Prior computational evidence	393
6.5	Geometrization	394
6.6	Hyperbolic knots and links	396
6.6.1	Knot complements	396
6.6.2	Link complements	397
6.7	Computation of hyperbolic manifolds	399
6.8	The Orbifold Theorem	401
6.9	Exercises and explorations	403
7	Line geometry	425
7.1	Half-rotations	425
7.2	The Lie product	426
7.3	Square roots	429
7.4	Complex distance	431
7.5	Complex distance and line geometry	432
7.6	Exercises and explorations	433
8	Right hexagons and hyperbolic trigonometry	444
8.1	Generic right hexagons	444
8.2	The sine and cosine laws	446
8.3	Degenerate right hexagons	448
8.4	Formulas for triangles, quadrilaterals, and pentagons	450
8.5	Exercises and explorations	453
	<i>Bibliography</i>	472
	<i>Index</i>	495