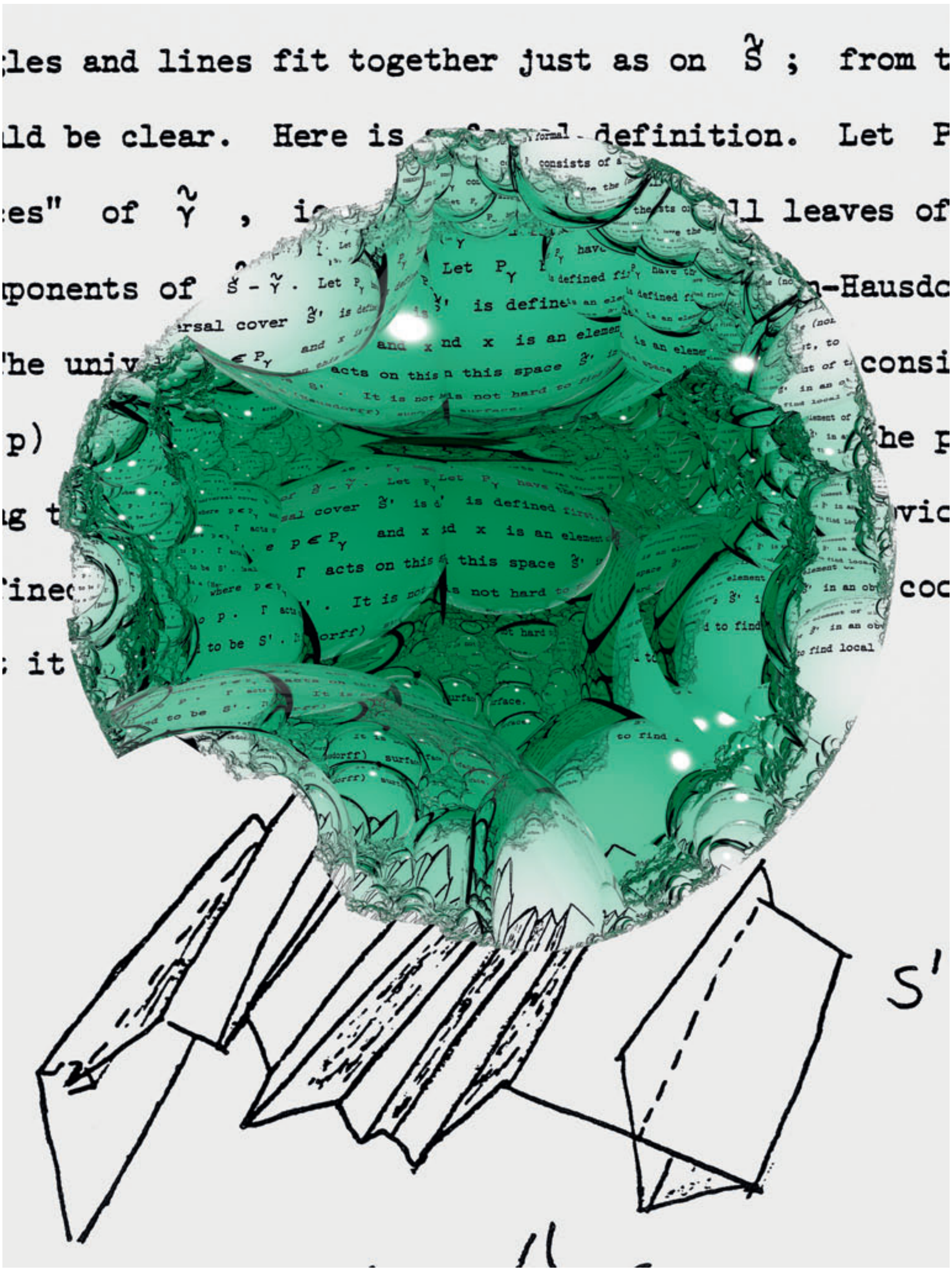
HYPERBOLIC MANIFOLDS

An Introduction in 2 and 3 Dimensions

Over the past three decades there has been a total revolution in the classic branch of mathematics called 3-dimensional topology, namely the discovery that most solid 3-dimensional shapes are hyperbolic 3-manifolds. This book introduces and explains hyperbolic geometry and hyperbolic 3- and 2-dimensional manifolds in the first two chapters, and then goes on to develop the subject. The author discusses the profound discoveries of the astonishing features of these 3-manifolds, helping the reader to understand them without going into long, detailed formal proofs. The book is heavily illustrated with pictures, mostly in color, that help explain the manifold properties described in the text. Each chapter ends with a set of Exercises and Explorations that both challenge the reader to prove assertions made in the text, and suggest further topics to explore that bring additional insight. There is an extensive index and bibliography.



[Thurston's Jewel (JB)(DD)] **Thurston's Jewel**: Illustrated is the convex hull of the limit set of a kleinian group G associated with a hyperbolic manifold $\mathcal{M}(G)$ with a single, incompressible boundary component. The translucent convex hull is pictured lying over p. 8.43 of Thurston [1979a] where the theory behind the construction of such convex hulls was first formulated. This particular hyperbolic manifold represents a critical point of Thurston's "skinning map", as described in Gaster [2012].

This image was created by Jeffrey Brock and David Dumas; details about its creation can be found at http://dumas.io/convex.

The image of p. 8.43 was used with permission of Julian, Nathaniel, Dylan Thurston.

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To Dorothy

צו מיין פרוי דבורה, די אמת'דיקע אשת-חיל.

and

In memory of William P. "Bill" Thurston
"God gave him the open book."—Jürgen Moser

Contents

List of Illustrations

Credits: CG, Charlie Gunn; CM, Curt McMullen; CS, Caroline Series; DD, David Dumas; DW, Dave Wright; HP, Howard Penner; JB, Jeff Brock; JP, John Parker; JR, Jonathan Rogness; KS, Ken Stephenson; RB, Robert Brooks; SL, Silvio Levy; StL, Stuart Levy; YM, Yair Minsky.

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Preface

To a topologist a teacup is the same as a bagel, but they are not the same to a geometer. By analogy, it is one thing to know the topology of a 3-manifold, another thing entirely to know its geometry—to find its shortest curves and their lengths, to make constructions with polyhedra, etc. In a word, we want to do geometry in the manifold just like we do geometry in euclidean space.

But do general 3-manifolds have "natural" metrics? For a start we might wonder when they carry one of the standards: the euclidean, spherical or hyperbolic metric. The latter is least known and not often taught; in the stream of mathematics it has always been something of an outlier. However it turns out that it is a big mistake to just ignore it! We now know that the interiors of "most" compact 3-manifolds carry a hyperbolic metric.

It is the purpose of this book to explain the geometry of hyperbolic manifolds. We will examine both the existence theory and the structure theory.

Why embark on such a study? Well after all, we do live in three dimensions; our brains are specifically wired to see well in space. It seems perfectly reasonable if not compelling to respond to the challenge of understanding the range of possibilities. For a while, it had even been considered that our own visual universe may be hyperbolic, although it is now believed that it is euclidean.

The twentieth-century history. Although Poincaré recognized in 1881 that Möbius transformations extend from the complex plane to upper half-space, the development of the theory of three-dimensional hyperbolic manifolds had to wait for progress in three-dimensional topology. It was as late as the mid-1950s that Papakyriakopoulos confirmed the validity of Dehn's Lemma and the Loop Theorem. Once that occurred, the wraps were off.

In the early 1960s, while 3-manifold topology was booming ahead, the theory of kleinian groups was abruptly awoken from its long somnolence by a brilliant discovery of Lars Ahlfors. Kleinian groups are the discrete isometry groups of hyperbolic 3-space. Working (as always) in the context of complex analysis, Ahlfors discovered their finiteness property. This was followed by Mostow's contrasting discovery that closed hyperbolic manifolds of dimension $n \geq 3$ are uniquely determined up to isometry by their isomorphism class. This too came as a bombshell as it is false for $n = 2$. Then came Bers' study of quasifuchsian groups and his and Maskit's fundamental discoveries of "degenerate groups" as limits of them. Along a different line, Jørgensen developed the methods for dealing with sequences of kleinian groups, recognizing the existence of two distinct kinds of convergence which he called "algebraic" and "geometric". He also discovered a key class of examples, namely hyperbolic 3-manifolds that fiber over the circle.

It wasn't until the late 1960s that 3-manifold topology was sufficiently understood, most directly by Waldhausen's work, and the fateful marriage of 3-manifold topology to the complex analysis of the group action on $\mathbb{S}^2$ occurred. The first application was to the classification and analysis of geometrically finite groups and their quotient manifolds.

During the 1960s and 1970s, Riley discovered a slew of faithful representations of knot and link groups in $\mathrm{PSL}(2, \mathbb{C})$. Although these were seen as curiosities at the time, his examples pressed further the question of just what class of 3-manifolds did the hyperbolic manifolds represent? Maskit had proposed using his combination theorems to construct all hyperbolic manifolds from elementary ones. Yet Peter Scott pointed out that the combinations that were then feasible would construct only a limited class of 3-manifolds.

So by the mid-1970s there was a nice theory, part complex analysis, part three-dimensional geometry and topology, part algebra. No-one had the slightest idea as to what the scope of the theory really was. Did kleinian groups represent a large class of manifolds, or only a small sporadic class?

The stage (but not the players) was ready for the dramatic entrance in the mid-1970s of Thurston. He arrived with a proof that the interior of "most" compact 3-manifolds has a hyperbolic structure. He brought with him an amazingly original, exotic, and very powerful set of topological/geometrical tools for exploring hyperbolic manifolds. The subject of two- and three-dimensional topology and geometry was never to be the same again.

This book. Having witnessed at first hand the transition from a special topic in complex analysis to a subject of broad significance and application in mathematics, it seemed appropriate to write a book to record and explain the transformation. My idea was to try to make the subject accessible to beginning graduate students with minimal specific prerequisites. Yet I wanted to leave students with more than a routine compendium of elementary facts. Rather I thought students should see the big picture, as if climbing a watchtower to overlook the forest. Each student should end his or her studies having a personal response to the timeless question: What is this good for?

With such thoughts in mind, I have tried to give a solid introduction and at the same time to provide a broad overview of the subject as it is today. In fact today, the subject has reached a certain maturity. The characterization of those compact manifolds whose interiors carry a hyperbolic structure is complete, the final step being provided by Perelman's confirmation of the Geometrization Conjecture. Attention turned to the analysis of the internal structure of hyperbolic manifolds assuming only a finitely generated fundamental group. The three big conjectures left over from the 1960s and 1970s have been solved: Tameness, Density, and classification of the ends (ideal boundary components). And recently, entirely new and quite surprising structures to be described below have been discovered. If one is willing to climb the watchtower, the view is quite remarkable.

It is a challenge to carry out the plan as outlined. The foundation of the subject rests on elements of three-dimensional topology, hyperbolic geometry, and modern complex analysis. None of these are regularly covered in courses at most places.

I have attempted to meet the challenge as follows. The presentation of the basic facts is fairly rigorous. These are included in the first four chapters, plus the optional Chapters 7 and 8. These chapters include crash courses in three-manifold topology, covering surfaces and manifolds, quasiconformal mappings, and Riemann surface theory. With the basic information under our belts, Chapters 5 and 6 (as well as parts of Chapters 3 and 4) are expository, without most proofs. The reader will find there both the Hyperbolization Theorem and the newly discovered structural properties of general hyperbolic manifolds.

At the end of each chapter is a long section titled "Exercises and Explorations". Some of these are genuine exercises and/or important additional information directly related to the material in the chapter. Others dig away a bit at the proofs of some of the theorems by introducing new tools they have required. Still others are included to point out various paths one can follow into the deeper forest and beauty spots one can find there. Thus there are not only capsule introductions to big fields like geometric group theory, but presentations of other more circumscribed topics that I (at least) find fascinating and relevant.

The nineteenth-century history. For the full story consult Jeremy Gray's new book, particularly Gray [2013, Ch. 3], for a comprehensive treatment of this period.

The history of non-Euclidean geometry in the early nineteenth century is fascinating because of a host of conflicted issues concerning axiom systems in geometry, and the nature of physical space [Gray 1986; 2002].

Jeremy Gray [2002] writes:

> Few topics are as elusive in the history of mathematics as Gauss's claim to be a, or even the, discoverer of Non-Euclidean geometry. Answers to this conundrum often depend on unspoken, even shifting, ideas about what it could mean to make such a discovery. ... [A]mbiguities in the theory of Fourier series can be productive in a way that a flawed presentation of a new geometry cannot be, because there is no instinctive set of judgments either way in the first case, but all manner of training, education, philosophy and belief stacked against the novelties in the second case.

Gray goes on to quote from Gauss's 1824 writings:

> ... the assumption that the angle sum is less than 180° leads to a geometry quite different from Euclid's, logically coherent, and one that I am entirely satisfied with. It depends on a constant, which is not given a priori. The larger the constant, the closer the geometry to Euclid's. ...

The theorems are paradoxical but not self-contradictory or illogical.... All my efforts to find a contradiction have failed, the only thing that our understanding finds contradictory is that, if the geometry were to be true, there would be an absolute (if unknown to us) measure of length a priori. ... As a joke I've even wished Euclidean geometry was not true, for then we would have an absolute measure of length a priori.

From his detailed study of the history, Gray's conclusion expressed in his Zurich lecture is that the birth of non-Euclidean geometry should be attributed to the independently written foundational papers of Lobachevsky in 1829 and Bólyai in 1832. As expressed in Milnor [1994, p. 246], those two were the first "with the courage to publish" accounts of the new theory. Still,

[f]or the first forty years or so of its history, the field of non-Euclidean geometry existed in a kind of limbo, divorced from the rest of mathematics, and without any firm foundation.

This state of affairs changed upon Beltrami's introduction in 1868 of the methods of differential geometry, working with constant curvature surfaces in general. He gave the first global description of what we now call hyperbolic space. See Gray [1986, p. 351], Milnor [1994, p. 246], Stillwell [1996, pp. 7–62].

It was Poincaré who brought two-dimensional hyperbolic geometry into the form we study today. He showed how it was relevant to topology, differential equations, and number theory. Again I quote Gray, in his translation of Poincaré's work of 1880 [Gray 1986, p. 268–9].

There is a direct connection between the preceding considerations and the non-Euclidean geometry of Lobachevskii. What indeed is a geometry? It is the study of a *group of operations* formed by the displacements one can apply to a figure without deforming it. In Euclidean geometry this group reduces to *rotations* and *translations*. In the pseudo-geometry of Lobachevskii it is more complicated... [Poincaré's emphasis].

As already mentioned, the first appearance of what we now call Poincaré's conformal model of non-Euclidean space was in his seminal 1881 paper on kleinian groups. He showed that the action of Möbius transformations in the plane had a natural extension to a conformal action in the upper half-space model.

Actually the names "fuchsian" and "kleinian" for the isometry groups of two- and three-dimensional space were attached by Poincaré. However Poincaré's choice more reflects his generosity of spirit toward Fuchs and Klein than the mathematical reality. Klein himself objected to the name "fuchsian". His objection in turn prompted Poincaré to introduce the name "kleinian" for the discontinuous groups that do not preserve a circle. The more apt name would perhaps have been "Poincaré groups" to cover both cases.

Recent history. A reader turning to this book may benefit by first consulting some of the fine elementary texts now available before diving directly into the theory of hyperbolic 3-manifolds. I will mention in particular: Jim Anderson's text [Anderson 2005], Frances Bonahon's text [Bonahon 2009], Jeff Weeks' text [Weeks 2002], as well as the classic book of W. Thurston [Thurston 1997]. Ours remains a tough subject to enter because of the range of knowledge involved—geometry, topology, analysis, algebra, number theory—-still, those who take the plunge find satisfaction in the subject's richness.

Within the past three years, our subject has gone well beyond the major accomplishments of the Thurston era, which include proofs of Tameness, Density, and the Ending Lamination, and, most importantly, Perleman's proof of the full Geometrization Conjecture. In March, 2012 was the dramatic announcement by Ian Agol of the solution of the Virtual Haken and Virtual Fibering Conjectures for hyperbolic manifolds. As the principal architects of the proof in drawing together new elements of hyperbolic geometry, cubical complexes, and geometric group theory, Ian Agol and Dani Wise shared the 2013 AMS Veblen prize.

Their proof required the Surface Subgroup Conjecture which had been confirmed to great acclaim by Kahn and Markovic a few years earlier. In addition, by the same method Kahn and Markovic confirmed another major longstanding question: the Ehrenpries Conjecture about Riemann surfaces. Jeremy Kahn and Vlad Markovic were awarded the 2012 Clay Prize for their accomplishments. Further remarkable consequences are duly reported in Chapter 6.

Mahan Mj resolved a different longstanding problem. He proved that connected limit sets of kleinian groups are locally connected. He proved this as a consequence of his existence proof of general "Cannon-Thurston maps". This work is discussed in Chapter 5.

The resolution of another important issue has been announced by Ken'ichi Ohtsuka and Teruhiko Soma with a paper in arXiv. They determined all possible geometric limits at quasifuchsian space boundaries. It too is discussed in Chapter 5.

Although the bibliography is extensive, it is hardly inclusive of all the papers that have contributed to the subject. Every signature accomplishment has been built on prior work of many others. I have tended to include references only to papers that are the most comprehensive and those which put down the last word. Upon referring to the referenced papers, one gets a better idea of the extent of the prior contributions culminating in a final result.

Acknowledgments. It is a great pleasure to thank the many people who have helped bring the book and its predecessor to fruition.

First I want to acknowledge the essential contributions of my friend and colleague Troels Jørgensen. Over more than 25 years we walked in the forest together discussing and admiring the landscape our studies revealed. In particular we discussed the "universal properties" of Chapter 3 for years, until it was too late to publish them. Chapters 7 and 8 are based on his private lectures.

David Wright kindly computed a number of limit sets of kleinian groups, some never before seen, others adapted from pictures created for *Indra's Pearls* [Mumford et al. 2002]. The extent of his contribution is evident from the list of figures. His pictures can be downloaded from www.okstate.edu/∼wrightd/Marden together with computational details. In addition, David Dumas was willing to share

his visualization of a Bers slice amidst the surrounding archipelago of discreteness components. Jeff Brock contributed his pictures of algebraic and geometric limits that originally appeared in Brock [2001b]; these too can be seen on www.math.brown.edu/∼brock. The presence of the many artfully crafted pictures is a tangible expression of the mathematical beauty of the subject.

Jeff Brock and David Dumas created the stunning frontispiece "Thurston's Jewel"; its intricate rendering required 1425 CPU-hours. My colleague Jon Rogness created two clarifying diagrams for the text. The full list of figures and their creators is on p. xi.

The list of those providing timely advice on mathematical issues includes Scot Adams, Ian Agol, Shinpei Baba, Ara Basmajian, Brian Bowditch, Jeff Brock, Dick Canary, Daryl Cooper, David Dumas, Cliff Earle, David Epstein, Ren Guo, Misha Kapovich, Sadayoshi Kojima, Sara Maloni, Vlad Markovic, Howie Masur, Yair Minsky, Mahan Mj, Ken'ici Ohshika, Peter Scott, and Saul Schleimer.

I could not have completed the book in the present form without the expert guidance and participation of Silvio Levy. Not only did he help with the mathematics, he formatted much of the LaTeX, and led me out of numerous LaTeX jams.

I want to acknowledge the institutional support from the Forschungsinstitut für Mathematik at ETH in Zurich, the Maths Research Center, University of Warwick, and not least, from my own department, the School of Mathematics of the University of Minnesota. In my semester course Math 8380, I was able to present a solid introduction and overview of the subject based on the main points in the first six chapters.

My editor at Cambridge University Press is David Tranah. One could not have asked for a more supportive person, a wise advisor in all the publication issues. I am grateful to Caroline Series for introducing him.

So here we are today, nearly 130 years after Poincaré and approaching 200 after the initial ferment of ideas of Gauss. We are witnessing a full flowering of the vision and struggle for understanding of the nineteenth-century masters. Still, the final word remains an elusive goal.

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