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CHAPTER ONE

INTRODUCTION

1.1 NONLINEAR SOLID DYNAMICS

The nonlinear analysis of the mechanical behavior of solid continua can be categorized in a number of ways. Solids may exhibit material or geometric nonlinearity. In the former the constitutive behavior, that is, the stress-strain relationship will be nonlinear and in the latter geometric changes, such as large rotations, affect the behavior. In many situations, such as metal forming, both occur simultaneously. A further category is whether the response of the solid to loading, be it forces or temperature, is dynamic or static, or in other words time dependent or not, and to be more precise whether inertial forces are relevant or can be ignored. In a previous text entitled *Nonlinear Solid Mechanics for Finite Element Analysis: Statics* the authors covered the fundamental nonlinear continuum mechanics necessary for the development of the equilibrium equations and their eventual solution using finite element discretization. The present text extends that development into the nonlinear dynamic realm and, whereas it is reasonably self-contained, it is useful to have an awareness of the material in the companion Statics text.

The dynamic response of solids may be linear or nonlinear. Linear response is generally associated with small deformation vibration behavior about an equilibrium position where geometrically nonlinear effects are normally insignificant, but not always, as in the case of a vibrating string in tension. Examples of dynamic behavior that can be considered in the linear regime are the vibration of buildings under moderate earthquakes which do not take the structure anywhere near its possible failure. Nonlinear dynamic behavior is characterized primarily by the presence of large rotation in addition to possible large strain. For example, satellites exhibit large rotation small strain behavior, whereas the collapse of, say, a building due to an extreme earthquake or a high speed impact situation is the result of both large rotation and large strain.

As the title implies, the solid continua are modeled in a discrete manner by an assemblage of three-dimensional finite elements for which formulations are developed to represent the geometric and material behavior due to the motion of the solid with respect to time as a result of various types of loading. In general the spatial representation of the body in terms of finite elements is similar between dynamic and static applications. The main difference emerges in relation to the introduction of inertial forces due to the mass and acceleration of the body and the presence of time as a key independent variable.

Time being the essential difference between static and dynamic behavior means that procedures need to be devised in order to progress the motion of the solid as time progresses. These procedures are generally known as *time-stepping schemes* whereby the motion, that is, primarily the velocity and thence the position, are discretized in time. This means that the motion between discrete time steps is approximated in some way. A number of such time-stepping schemes are presented in this text, from the simple leap-frog scheme to more complex and general Newmark schemes.

Fundamental to the description of dynamic behavior, linear or nonlinear, is Newton's Second Law of Motion, giving a dynamic equilibrium equation relating force, mass, and acceleration. Motion is progressed by determining the acceleration and then using the time-stepping scheme to find the velocity and advance the position. In mathematical terms this implies the solution of a second-order equation in time by an appropriate time-stepping scheme.

An alternative approach presented in this text involves the reformulation of the dynamic equilibrium of a solid into a system of first-order conservation laws for the physical and geometric variables describing the solid and its motion. This process leads to a mixed set of unknowns incorporating both velocities and strains. Similar sets of conservation laws are used extensively in computational fluid dynamics and the discretization of such laws via the finite element method is well understood and routinely applied. In this text both the traditional displacement-based approach, leading to a second-order system of equations in time, and the first-order set of conservation laws will be presented to solve solid dynamics problems. In the final chapters a number of examples will be solved using both these approaches to demonstrate their validity and general applicability.

The general aim of this book is to provide the reader with a good understanding of the necessary continuum mechanics concepts and theory required to successfully model by finite elements the time-dependent large deformation of solids, including possible thermal effects. It will cover important continuum concepts such as virtual work, potential and kinetic energy, the Lagrangian, and Hamilton's principle, as well as thermodynamics and thermoelasticity. It will also present time-discretization issues such as energy and momentum conserving schemes and simplectic integrators or advanced finite element discretization technologies such as Petrov–Galerkin methods for first-order conservation laws. Although the authors have made every effort to keep the text self-contained, the reader would benefit from some degree of familiarity with the contents of the companion statics volume: *Nonlinear Solid Mechanics for Finite Element Analysis: Statics*. This text will be referred to in the remainder of this book as the NL-Statics Volume.

The remainder of this chapter sets out to provide a gentle introduction to nonlinear dynamic behavior via simple one- or two-degrees-of-freedom examples. These examples are used to introduce simple time integration schemes such as the leap-frog method or the mid-point rule and discuss important issues associated with these schemes such as stability, or to compare implicit versus explicit methodologies.

1.2 ONE-DEGREE-OF-FREEDOM NONLINEAR DYNAMIC BEHAVIOR

In this section the torsion spring supported single rigid column discussed in Section 1.2.2 of NL-Statics is re-examined to demonstrate a simple example of nonlinear dynamic behavior. In order to introduce inertial forces, the column supports a mass which is restrained by a linear elastic torsion spring and viscous torsion damping at the bottom hinge, see Figure 1.1. The governing equations do not admit an analytical solution and consequently a simple numerical "leap-frog" integration in time will be introduced in order to obtain the behavior of the column with respect to time.

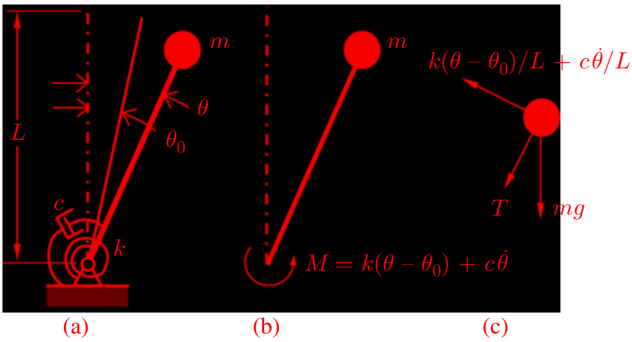

FIGURE 1.1 Simple column.

1.2.1 Equation of Motion

The column shown in Figure 1.1, where the angular motion of the mass m, from an initial angle of θ_0, is constrained by a torsion spring having stiffness k and, in parallel, torsional viscous damping having a coefficient c. The effect of introducing viscosity into the system will be demonstrated below. The torsion spring produces a moment $k(\theta-\theta_0)$ which in turn is equivalent to a tangential force on the mass equal to $k(\theta - \theta_0)/L$. The viscous damping produces a similar force but now equal to $c\,\dot{\theta}/L$, where $\dot{\theta}$ is the angular velocity. Elementary kinematics gives the tangential and radial acceleration of the mass m in terms of $\dot{\theta}$ and the tangential acceleration $\ddot{\theta}$ as $L\ddot{\theta}$ and $L\dot{\theta}^2$, respectively.

Employing Newton's Second Law of Motion the dynamic equilibrium equations in the tangential and radial directions are

$$mL\frac{d^2\theta}{dt^2} = mg\sin\theta - \frac{c}{L}\frac{d\theta}{dt} - \frac{k}{L}(\theta - \theta_0), \tag{1.1a}$$

$$mL\left(\frac{d\theta}{dt}\right)^2 = T + mg\cos\theta. \tag{1.1b}$$

Observe that the first of these equations is sufficient to determine the motion of the column via the evaluation of $\theta(t)$, whereas the second equation enables the calculation of the tension, T, in the column once the angle $\theta(t)$ and angular velocity $\dot{\theta}(t)$ have been obtained. The geometric nonlinearity in, for example, the first of the above equations is enshrined in the tangential gravitational force term $mg\sin\theta$.

As mentioned above, unlike the static column example considered in NL-Statics Section 1.2.2, the above equations do not readily admit an analytical solution and have to be solved using a numerical time integration scheme which enables the angular velocity $\dot{\theta}(t)$ and hence the angular position $\theta(t)$ to be calculated after the acceleration $\ddot{\theta}(t)$ has been found from Equation (1.1a). Numerous time integration schemes exist and a number of these will be introduced in this book, but for now a simple robust scheme known as leap-frog time integration will be employed.

1.2.2 Leap-Frog Time Integration

The column dynamic tangential equilibrium Equation (1.1a) can be rewritten to give the tangential acceleration as

$$\mathbf{a} = \frac{g}{L}\sin(\mathbf{x}) - \frac{c}{mL^2}\mathbf{v} - \frac{k}{mL^2}(\mathbf{x} - \mathbf{X}); \quad \mathbf{a} = \frac{d^2\theta}{dt^2}; \quad \mathbf{v} = \frac{d\theta}{dt}; \quad \mathbf{x} = \theta, \tag{1.2}$$

where the notation for acceleration $\mathbf{a}$, velocity $\mathbf{v}$, and coordinate position $\mathbf{x}$ provide for a general description of the time-stepping scheme* that will be valid for multiple-degree-of-freedom problems.

* For the column problem $\mathbf{x} = \theta$, $\mathbf{v} = \dot{\theta}$, and $\mathbf{X} = \theta_0$.

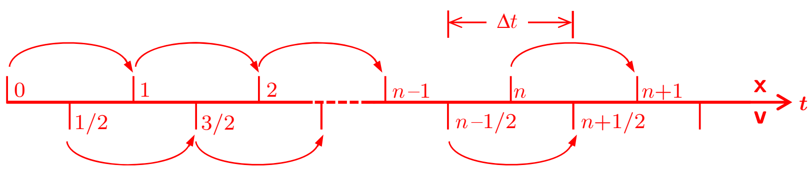

FIGURE 1.2 Leap-frog time integration.

The time during which the motion takes place is now divided into a number of equal time steps Δt, where a typical time is labeled n, see Figure 1.2. Using the above acceleration $\mathbf{a}$, assuming $\mathbf{v}$ and $\mathbf{x}$ to be known, an approximate integration over a time step Δt can be employed to calculate the velocity at the half-time-step $t_{n+1/2}$ in terms of the previous half time step velocity and the time step $\Delta t = t_{n+1/2} - t_{n-1/2}$ as

$$\mathbf{v}_{n+1/2} = \mathbf{v}_{n-1/2} + \int_{t_{n-1/2}}^{t_{n+1/2}} \mathbf{a}\, dt \approx \mathbf{v}_{n-1/2} + \mathbf{a}_n \Delta t. \tag{1.3}$$

Similarly the updated coordinates $\mathbf{x}_{n+1}$ are found using the velocity at the previous half time step above to yield

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \int_{t_n}^{t_{n+1}} \mathbf{v}\, dt \approx \mathbf{x}_n + \mathbf{v}_{n+1/2} \Delta t. \tag{1.4}$$

This process, illustrated in Figure 1.2, is an example of a staggered time-stepping scheme.

Notice in Figure 1.2 that at the first time step $n = 0$ the velocity at $\mathbf{v}_{1/2}$ is required to start the process. Given an initial velocity $\mathbf{v}_0$ and an initial acceleration $\mathbf{a}_0$ calculated from Equation (1.2), $\mathbf{v}_{1/2}$ can be found using the approximation

$$\mathbf{v}_{1/2} = \mathbf{v}_0 + \mathbf{a}_0 \frac{\Delta t}{2}. \tag{1.5}$$

Observe that, given the staggered nature of the scheme, unless the viscosity vanishes, that is $c = 0$, the velocity in Equation (1.2) is not generally known at time n and needs to be taken at $t_{n-1/2}$ in order to allow a direct or explicit evaluation of the acceleration, $\mathbf{a}_n$, at time n as

$$\mathbf{a}_n = \frac{g}{L}\sin(\mathbf{x}_n) - \frac{c}{mL^2}\mathbf{v}_{n-1/2} - \frac{k}{mL^2}(\mathbf{x}_n - \mathbf{X}). \tag{1.6}$$

The exception to this rule occurs at $n = 0$, when $\mathbf{v}_0$ is actually known from the initial conditions, and hence $\mathbf{a}_0$ can be evaluated as†

$$\mathbf{a}_0 = \frac{g}{L}\sin(\mathbf{x}_0) - \frac{c}{mL^2}\mathbf{v}_0 - \frac{k}{mL^2}(\mathbf{x}_0 - \mathbf{X}). \tag{1.7}$$

† $\mathbf{x}_0$ need not be equal to ; see Exercise 5 of Chapter 2.

It is clear that the replacement of $\mathbf{v}_n$ by $\mathbf{v}_{n-1/2}$ in Equation (1.6) introduces an error in the calculation of $\mathbf{a}_n$. This can be avoided by using the correct $\mathbf{v}_n$ expressed as $\mathbf{v}_n = \mathbf{v}_{n-1/2} + (\Delta t/2)\mathbf{a}_n$ to give an expression for $\mathbf{a}_n$ as

$$\mathbf{a}_n = \frac{g}{L}\sin(\mathbf{x}_n) - \frac{c}{mL^2}\left(\mathbf{v}_{n-1/2} + \frac{\Delta t}{2}\mathbf{a}_n\right) - \frac{k}{mL^2}(\mathbf{x}_n - \mathbf{X}). \tag{1.8}$$

Consequently, $\mathbf{a}_n$ is now implicit in its own evaluation. For a one-degree-of-freedom problem this is not a problem as it would suffice to move all terms containing $\mathbf{a}_n$ to the left of the equation and divide the remaining right side by the accumulated coefficient of $\mathbf{a}_n$, to yield

$$\mathbf{a}_n = \left(\frac{g}{L}\sin(\mathbf{x}_n) - \frac{c}{mL^2}\mathbf{v}_{n-1/2} - \frac{k}{mL^2}(\mathbf{x}_n - \mathbf{X})\right)\Big/\Big(1 + \frac{c}{mL^2}\frac{\Delta t}{2}\Big). \tag{1.9}$$

Whereas Equation (1.6) is an explicit equation, Equation (1.8) is an implicit equation requiring a solution for $\mathbf{a}_n$. In the single-degree-of-freedom case, solving Equation (1.9) is trivial, but for realistic simulations containing large numbers of degrees of freedom k, c, and m are matrices, which leads to a system of equations requiring solution. Generally, time-stepping schemes in which the variables can be updated without solving systems of equations are known as *explicit*, whereas schemes that require the solution of a system of equations are known as *implicit*, and if this is required at every time step such schemes are computationally costly.

In comparison to an implicit scheme, the greater efficiency of an explicit scheme would seem preferable; however, it will be seen later that they suffer from severe limitations with respect to time-step size before producing grossly inaccurate solutions. These time-step limitations are known as stability restrictions and will be discussed in Section 1.3.3 below. Implicit time-stepping schemes are usually constructed in such a manner that avoids stability restrictions and consequently allow much larger time steps to be used, albeit at greater cost per step. The leap-frog algorithm is shown in Box 1.1. This explicit algorithm is of general applicability and as a consequence does not include the correction implied by Equation (1.9).

BOX 1.1: Leap-frog algorithm

- INPUT geometry, material properties, and solution parameters
- INITIALIZE $\mathbf{x}_0$, $\mathbf{v}_0$
- SET $\mathbf{a}_0$ (1.7) using initial values
- SET $\mathbf{v}_{1/2}$ (1.5)
- DO WHILE $t < tmax$ (time steps)
  - SET $\mathbf{x} = \mathbf{x} + \mathbf{v}\,\Delta t$ (1.4)
  - FIND $\mathbf{a}$ (1.6)

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Box 1.1: *(cont.)*

- SET $\mathbf{v} = \mathbf{v} + \mathbf{a}\,\Delta t$ (1.3)
- ENDDO

1.2.3 Column Examples

The following examples employ the leap-frog time integration discussed in Section 1.2.2. In Figure 1.3 the viscous coefficient is $c = 0$ and in Figure 1.4 the viscous coefficient is $c = 650$.‡ Both examples have an initial angle of $\theta_0 = 45°$, $L = 10$, $m = 100$, $g = 9.81$, $k = 1000$, $\Delta t = 0.0001$, and the initial tangential velocity is $\mathbf{v}_0 = 0$. Allowing the time to extend to $t = 200$ reveals that the damped solution converges to the static solution of $\theta \approx 167.42°$.

1.3 TWO-DEGREES-OF-FREEDOM EXAMPLE

The spring-mass system shown in Figure 1.5 will be used to introduce the effects of geometric nonlinearity in a dynamic situation. The internal force in the spring is $\mathbf{T}$ and the spring has a stiffness k. At time $t = 0$ the orientation of the spring is

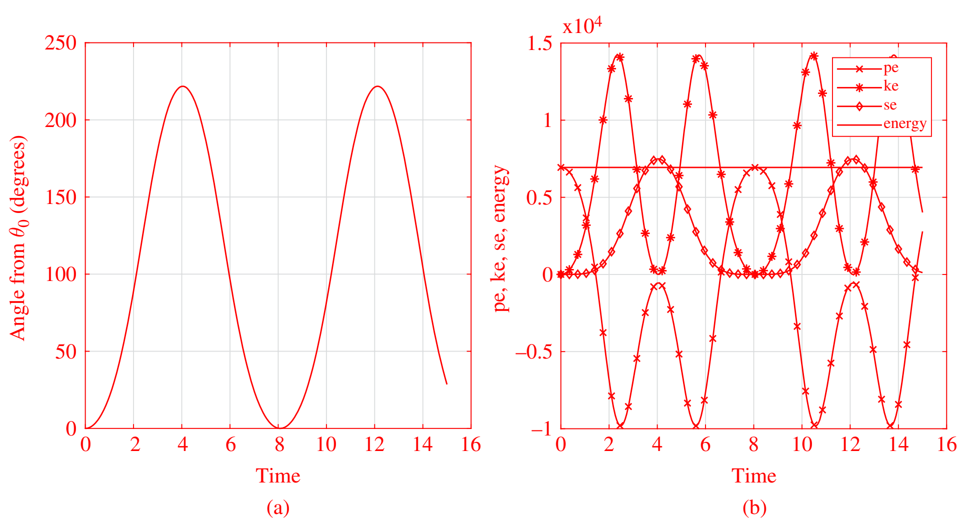

FIGURE 1.3 Column leap-frog with $\theta_0 = 45°, c = 0$: (a) Angle from θ_0–time; (b) Energy–time (pe = potential energy, se = elastic (strain) energy, ke = kinetic energy, total energy).

‡ For this case the correction for a_n given by the last term in Equation (1.9) involving the viscosity c is negligible and is not employed in the solution.

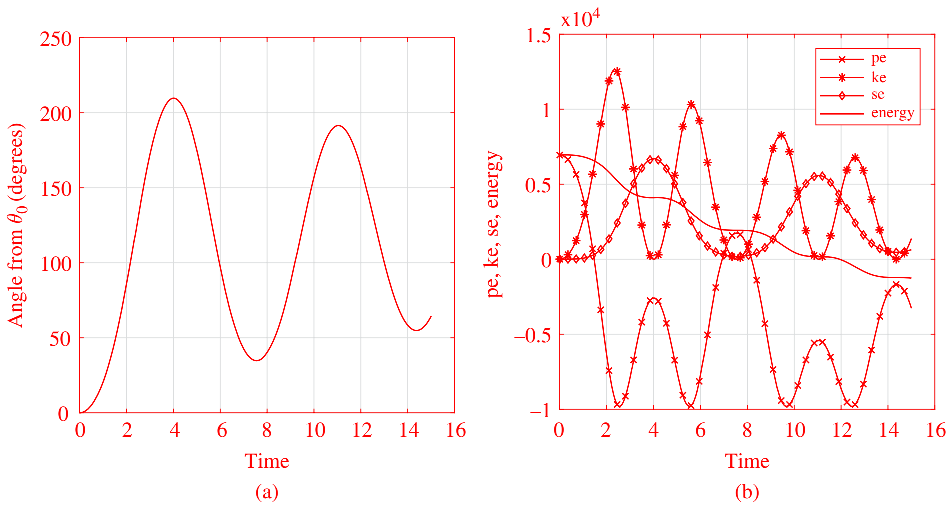

FIGURE 1.4 Column leap-frog with $\theta_0 = 45°, c = 650$: (a) Angle from θ_0–time; (b) Energy–time.

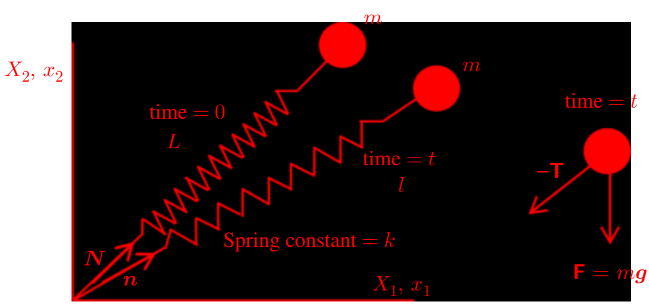

FIGURE 1.5 Two-degrees-of-freedom example.

given by the unit normal $\boldsymbol{N}$ and the length is L, while at time t the orientation is given by $\boldsymbol{n}(t)$ and the length is $l(t)$. The unit vectors $\boldsymbol{N}$ and $\boldsymbol{n}(t)$ at time $t = 0$ and t respectively are determined by the corresponding mass coordinates $\mathbf{X}$ and $\mathbf{x}(t)$ as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}; \quad \boldsymbol{N} = \frac{1}{L}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \tag{1.10a,b}$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad \boldsymbol{n}(t) = \frac{1}{l}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad l = \sqrt{(\mathbf{x} \cdot \mathbf{x})}. \tag{1.10c,d,e}$$

Time derivatives of the coordinate $\mathbf{x}(t)$ give the velocity and acceleration as

$$\mathbf{v}(t) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}; \quad \mathbf{v}(t) = \frac{d\mathbf{x}}{dt}; \quad \mathbf{a}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}; \quad \mathbf{a}(t) = \frac{d^2\mathbf{x}}{dt^2}. \tag{1.11a,b,c,d}$$

For simplicity the explicit dependency of quantities upon time will be removed, for example, $\mathbf{x}(t)$ becomes $\mathbf{x}$, etc. The internal spring force $\mathbf{T}$ and the vertical gravitational force $\mathbf{F}$ are

$$\mathbf{T}(\mathbf{x}) = T\boldsymbol{n}; \quad T = k(l - L); \quad \mathbf{F} = \begin{bmatrix} 0 \\ -mg \end{bmatrix}. \tag{1.12a,b}$$

Obviously, on Earth at least, $g = 9.81\,\text{m/s}^2$.

1.3.1 Equations of Motion

Employing Newton's Second Law of Motion applied to the mass m at time t, the equations of motion are written in terms of the acceleration $\mathbf{a}$, internal force $\mathbf{T}$, and external force $\mathbf{F}$ as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \end{bmatrix} - \frac{T}{l} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ or simply } \mathbf{Ma} = \mathbf{F} - \mathbf{T}(\mathbf{x}). \tag{1.13a,b}$$

Equation (1.12a,b) reveals that the internal force $\mathbf{T}$ at time t is a function of the length l and unit normal $\boldsymbol{n}$, both being functions of the current position $\mathbf{x}$ of the mass. Consequently, the equations of motion (1.13a,b) are geometrically nonlinear. Such equations do not admit an analytical solution and have to be solved numerically in the following section using the leap-frog scheme.

EXAMPLE 1.1: Energy conservation

The two-degrees-of-freedom spring-mass system described in this section is a convenient example with which to illustrate the conservation of physical quantities such as energy. The equilibrium Equation (1.13a,b) for the mass at time t and position $\mathbf{x}$ can be rewritten as

$$m\mathbf{a} + k(l - L)\boldsymbol{n} = m\boldsymbol{g},$$

where

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}; \; \boldsymbol{n} = \frac{\mathbf{x}}{l}; \; l = \|\mathbf{x}\|; \; \boldsymbol{g} = -g \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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Example 1.1: *(cont.)*

Multiplying the equilibrium equation by $\mathbf{v}$ and rearranging gives

$$m\mathbf{a}\cdot\mathbf{v} + k(l-L)\boldsymbol{n}\cdot\mathbf{v} + mg\,v_2 = 0,$$

noting that

$$\mathbf{a}\cdot\mathbf{v} = \frac{1}{2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}\frac{dv^2}{dt};\; v = \|\mathbf{v}\|;\; v_2 = \frac{dx_2}{dt},$$

and

$$\boldsymbol{n}\cdot\mathbf{v} = \frac{\mathbf{x}}{l}\cdot\frac{d\mathbf{x}}{dt} = \frac{1}{2}\frac{1}{l}\frac{d}{dt}(\mathbf{x}\cdot\mathbf{x}) = \frac{1}{2}\frac{1}{l}\frac{dl^2}{dt} = \frac{dl}{dt}$$

gives, after some simple algebra,

$$\frac{d}{dt}\left[\frac{1}{2}mv^2 + \frac{1}{2}k(l-L)^2 + mgx_2\right] = 0.$$

The term in the square bracket is the total energy of the mass m comprising the kinetic, elastic, and potential components, thus demonstrating the conservation of total energy.

1.3.2 Leap-Frog Examples

Using the two-degrees-of-freedom formulation, the column problem given in Section 1.2.3 can be rerun with a high linear spring constant of $k = 10^5$ to approximate a rigid column. This is equivalent to the column case shown in Figure 1.1 with a torsion spring constant of value zero. Figure 1.6 shows the results for the two-degrees-of-freedom simulation. When comparing with Figure 1.3, note that due to the presence of the torsion spring in Figure 1.3 the maximum angle is, as expected, less than that given in Figure 1.6 where the torsion spring stiffness is necessarily zero.

The next example is essentially a simple linear spring-mass problem with one degree of freedom. In order to achieve this, the mass is constrained to move along a fixed axis, as shown in Figure 1.7 (see also Exercise 2). This example will show that for a stable implementation of the leap-frog solution the value of the time step must be such that $\Delta t < 2/\omega$ where $\omega = (k/m)^{\frac{1}{2}}$. For a mass of unity and a spring stiffness $k = 1000$, Figure 1.8 shows the leap-frog solutions for time step $\Delta t = 0.01/\omega$ (solid plot) and $\Delta t = 2.01/\omega$ (dashed plot). For $\Delta t < 2/\omega$ the solution is identical to that given analytically in Exercise 2, having a maximum amplitude of $(1-\cos(\omega t)) = 2$, whereas for $\Delta t > 2/\omega$ the numerical solution is clearly unstable. More interesting examples will be reserved for Section 1.3.6.