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Optimal Control and Geometry: Integrable Systems

The synthesis of symplectic geometry, the calculus of variations, and control theory offered in this book provides a crucial foundation for the understanding of many problems in applied mathematics.

Focusing on the theory of integrable systems, this book introduces a class of optimal control problems on Lie groups whose Hamiltonians, obtained through the Maximum Principle of optimality, shed new light on the theory of integrable systems. These Hamiltonians provide an original and unified account of the existing theory of integrable systems. The book particularly explains much of the mystery surrounding the Kepler problem, the Jacobi problem, and the Kowalewski Top. It also reveals the ubiquitous presence of elastic curves in integrable systems up to the soliton solutions of the non-linear Schroedinger's equation.

Containing a useful blend of theory and applications, this is an indispensable guide for graduates and researchers, in many fields from mathematical physics to space control.

PROFESSOR JURDJEVIC is one of the founders of geometric control theory. His pioneering work with H. J. Sussmann was deemed to be among the most influential papers of the century and his book, *Geometric Control Theory*, revealed the geometric origins of the subject and uncovered important connections to physics and geometry. It remains a major reference on non-linear control. Professor Jurdjevic's expertise also extends to differential geometry, mechanics and integrable systems. His publications cover a wide range of topics including stability theory, Hamiltonian systems on Lie groups, and integrable systems. He has spent most of his professional career at the University of Toronto.

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Optimal Control and Geometry: Integrable Systems

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Acknowledgments

This book grew out of the lecture notes written during the graduate courses that I gave at the University of Toronto in the mid 2000s. These courses made me aware of the need to bridge the gap between mainstream mathematics, differential geometry, and integrable systems, and control theory, and this realization motivated the initial conception of the book.

I am grateful to the University of Toronto for imposing the mandatory retirement that freed my time to carry out the necessary research required for the completion of this project.

There are several conferences and workshops which have had an impact on this work. In particular I would like to single out the Conference of Geometry, Dynamics and Integrable systems, first held in 2010 in Belgrade, Serbia and second in Sintra – Lisbon, Portugal in 2011, the INDAM meeting on Geometric Control and sub-Riemannian Geometry, held in Cortona, Italy in 2012, the IV Ibaronamerican Meeting on Geometric Mechanics and Control held in Rio de Janeiro in 2014, and the IMECC/Unicamp Fourth School and Workshop on Lie theory held in Campinas, Brazil in 2015. I thank the organizers for giving me a chance to present some of the material in the book and benefit from the interaction with the scientific community.

I also would like to thank the editorial staff of Cambridge University Press for providing an invaluable assistance in transforming the original manuscript to its present form. In particular, my thanks go to David Hemsley for his painstaking work in unraveling the inconsistencies and ambiguities in my original submission.

I feel especially indebted to my wife Ann Bia for taking care of the life around us during my preoccupation with this project. Thank you Ann.

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Introduction

Upon the completion of my book on geometric control theory, I realized that this subject matter, which was traditionally regarded as a domain of applied mathematics connected with the problems of engineering, made important contributions to mathematics beyond the boundaries of its original intent. The fundamental questions of space control, starting with the possibility of navigating a dynamical system from an initial state to a given final state, all the way to finding the best path of transfer, inspired an original theory of differential systems based on Lie theoretic methods, and the quest for the best path led to the Maximum Principle of optimality. This theory, apart from its relevance for the subject within which it was conceived, infuses the calculus of variations with new and fresh insights: controllability theory provides information about the existence of optimal solutions and the Maximum Principle leads to the solutions via the appropriate Hamiltonians. The new subject, a synthesis of the calculus of variations, modern symplectic geometry and control theory, provides a rich foundation indispensable for problems of applied mathematics.

This recognition forms the philosophical underpinning for the book. The bias towards control theoretic interpretations of variational problems provides a direct path to Hamiltonian systems and reorients our understanding of Hamiltonian systems inherited from the classical calculus of variations in which the Euler–Lagrange equation was the focal point of the subject. This bias also reveals a much wider relevance of Hamiltonian systems for problems of geometry and applied mathematics than previously understood, and, at the same time, it offers a distinctive look at the theory of integrable Hamiltonian systems.

This book is inspired by several mathematical discoveries in the theory of integrable systems. The starting point was the discovery that the mathematical formalism initiated by G. Kirchhoff to model the equilibrium configurations of a thin elastic bar subjected to twisting and bending torques at its ends can be

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reformulated as an optimal control problem on the orthonormal frame bundle of $\mathbb{R}^3$, with obvious generalizations to any Riemannian manifold. On three-dimensional spaces of constant curvature, where the orthonormal frame bundle coincides with the isometry group, this generalization of Kirchhoff's elastic model led to a left-invariant Hamiltonian H on a six-dimensional Lie group whose Hamiltonian equations on the Lie algebra showed remarkable similarity with the equations of motion for the heavy top (a rigid body fixed at a point and free to move around this point under the gravitational force). Further study revealed an even more astonishing fact, that the associated control Hamiltonian system is integrable precisely in three cases under the same conditions as the the heavy top [JA; Jm].

This discovery showed that the equations of the heavy top form an invariant subsystem of the above Hamiltonian system and that the solvability of the equations for the top is subordinate to the integrability of this Hamiltonian system (and not the other way around as suggested by Kirchhoff and his "kinetic analogue" metaphor [Lv]. More importantly, this discovery suggested that integrability of mechanical tops is better understood throgh certain left-invariant Hamiltonians on Lie groups, rather than through conventional methods within the confines of Newtonian physics.

The above discovery drew attention to a larger class of optimal control problems on Lie groups G whose Lie algebra $\mathfrak{g}$ admits a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ subject to

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, [\mathfrak{p}, \mathfrak{k}] = \mathfrak{p}, [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}. \tag{I.1}$$

These optimal problems, defined by an element $A \in \mathfrak{p}$ and a positive definite quadratic form $\langle\, ,\, \rangle$ on $\mathfrak{k}$, consist of finding the solutions of the affine control system

$$\frac{dg}{dt} = g(A + U(t)), U(t) \in \mathfrak{k}, \tag{I.2}$$

that conform to the given boundary conditions $g(0) = g_0$ and $g(T) = g_1$, for which the integral $\frac{1}{2}\int_0^T \langle U(t), U(t)\rangle\, dt$ is minimal. This class of optimal control problems is called *affine-quadratic*. We show that any affine-quadratic problem is well defined for any regular element A in $\mathfrak{p}$ in the sense that for any any pair of points g_0 and g_1 in G, there exists a time $T > 0$, and a control $U(t)$ on $[0, T]$ that generates a solution $g(t)$ in (2) with $g(0) = g_0$ and $g(T) = g_1$, and attains the minimum of $\frac{1}{2}\int_0^T \langle U(t), U(t)\rangle\, dt$.

Remarkably, the Hamiltonians associated with these optimal problems reveal profound connections with integrable systems. Not only do they link mechanical tops with geodesic and elastic problems, but also reveal the hidden

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symmetries, even for the most enigmatic systems such as Jacobi's geodesic problem on the ellipsoid, and the top of Kowalewski.

This book lays out the mathematical foundation from which these phenomena can be seen in a unified manner. As L. C. Young notes in his classical book on the calculus of variations and optimal control [Yg], problems of optimality are not the problems to tackle with bare hands, but only when one is properly equipped. In the process of preparing ourselves for the tasks ahead it became necessary to amalgamate symplectic and Poisson geometry with control theory. This synthesis forms the theoretic background for problems of optimality. Along the way, however, we discovered that this theoretic foundation also applies to the classic theory of Lie groups and symmetric spaces as well. As a result, the book turned out to be as much about Lie groups and homogeneous spaces, as is about the problems of the calculus of variations and optimal control.

The subject matter is introduced through the basic notions of differential geometry, manifolds, vector fields, differential forms and Lie brackets. The first two chapters deal with the accessibility theory based on Lie theoretic methods, an abridged version of the material presented earlier in [Jc]. The orbit theorem of this chapter makes a natural segue to the chapters on Lie groups and Poisson manifolds, where it is used to prove that a closed subgroup of a Lie group is a Lie group and that a Poisson manifold is foliated by symplectic manifolds. The latter result is then used to show that the dual of a Lie algebra is a Poisson manifold, with its Poisson structure inherited from the symplectic structure of the cotangent bundle, in which the symplectic leaves are the coadjoint orbits. This material ends with a discussion of left-invariant Hamiltonians, a prelude to the Maximum Principle and differential systems with symmetries.

The chapter on the Maximum Principle explains the role of optimal control for problems of the calculus of variations and provides a natural transition to the second part of the book on integrable systems. The Maximum Principle is presented through its natural topological property as a necessary condition for a trajectory to be on the boundary of the reachable set. The topological view of this principle allows for its strong formulation over an enlarged system, called the Lie saturate, that includes all the symmetries of the system. This version of the Maximum Principle is called the Saturated Maximum Principle. It is then shown that Noether's theorem and the related Moment map associated with the symmetries are natural consequences of the Saturated Maximum Principle.

This material forms the theoretic background for the second part of the book, which, for the most part, deals with specific problems. This material begins with a presentation of the non-Euclidean geometry from the Hamiltonian point

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of view. This choice of presentation illustrates the relevance of the above formalism for the problems of geometry and also serves as the natural segue to the chapter on Lie groups G with an involutive automorphism σ and to the geometric problems on G induced by the associated Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ of the Lie algebra $\mathfrak{g}$ of G. In these situations the Cartan decomposition then yields a splitting $\mathcal{F}_\mathfrak{p} \oplus \mathcal{F}_\mathfrak{k}$ of the tangent bundle TG, with $\mathcal{F}_\mathfrak{p}$ and $\mathcal{F}_\mathfrak{k}$ the families of left-invariant vector fields on G that take values in $\mathfrak{p}$, respectively $\mathfrak{k}$, at the group identity e. The distributions defined by these families of vector fields, called vertical and horizontal form a basis for the class of variational problems on G described below.

Vertical distribution $\mathcal{F}_\mathfrak{k}$ is involutive and its orbit through the group identity e is a connected Lie subgroup K of G whose Lie algebra is $\mathfrak{k}$. This subgroup is contained in the set of fixed points of σ and is the smallest Lie subgroup of G with Lie algebra equal to $\mathfrak{k}$, and can be regarded as the structure group for the homogeneous space $M = G/K$.

Horizontal family $\mathcal{F}_\mathfrak{p}$ is in general not involutive. We then use the Orbit theorem to show that on semi-simple Lie algebras the orbit of $\mathcal{F}_\mathfrak{p}$ through the group identity is equal to G if and only if $[\mathcal{F}_\mathfrak{p}, \mathcal{F}_\mathfrak{p}] = \mathcal{F}_\mathfrak{k}$, or, equivalently, if and only if $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. This controllability condition, translated to the language of the principal bundles, says that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ is a necessary and sufficient condition that any two points in G can be connected by a horizontal curve in G, where a horizontal curve is a curve that is tangent to $\mathcal{F}_\mathfrak{p}$.

The aforementioned class of problems on G is divided into two classes each treated somewhat separately. The first class of problems, inspired by the Riemannian problem on $M = G/K$ defined by a positive-definite, Ad_K-invariant quadratic form $\langle\, , \rangle$ on $\mathfrak{p}$ is treated in Chapter 8. In contrast to the existing literature on symmetric spaces, which introduces this subject matter through the geodesic symmetries of the underlying symmetric space [Eb; Hl], the present exposition is based on the pioneering work of R. W. Brockett [Br1; Br2] and begins with the sub-Riemannian problem of finding a horizontal curve $g(t)$ in G of minimal length $\int_0^T \sqrt{\left\langle g^{-1}(t)\frac{dg}{dt}, g^{-1}(t)\frac{dg}{dt}\right\rangle}\, dt$ that connects given points g_0 and g_1 under the assumption that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$.

We demonstrate that this intrinsic sub-Riemannian problem is fundamental for the geometry of the underlying Riemannian symmetric space G/K, in the sense that all of its geometric properties can be extracted from $\mathfrak{g}$, without ever descending onto the quotient space G/K. We show that the associated Hamiltonian system is completely integrable and that its solutions can be written in closed form as

$$g(t) = g_0 \exp t(A + B) \exp(-tB), A \in \mathfrak{p}, B \in \mathfrak{h}. \tag{I.3}$$

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The projection of these curves on the underlying manifold G/K coincides with the curves of constant geodesic curvature, with $B = 0$ resulting in the geodesics. We then extract the Riemannian curvature tensor

$$\kappa(A, B) = \langle [[A, B], A], B \rangle, A \in \mathfrak{p},\ B \in \mathfrak{p}. \tag{I.4}$$

from the associated Jacobi equation. The chapter ends with a detailed analysis of the Lie algebras associated with symmetric spaces of constant curvature, the setting frequently used in the rest of the text.

The second aforementioned class of problems, called affine-quadratic, presented in Chapter 9, in a sense is complementary to the sub-Riemannian case mentioned above, and is most naturally introduced in the language of control theory as an optimal control problem over an affine distribution $\mathcal{D}(g) = \{g(A+U) : U \in \mathfrak{k}$ defined by an element A in $\mathfrak{p}$ and a positive-definite quadratic form $Q(u, v)$ defined on $\mathfrak{k}$. The first part deals with controllability, as a first step to the well-posedness of the problem. We first note a remarkable fact that any semi-simple Lie algebra $\mathfrak{g}$, as a vector space, carries two Lie bracket structures: the semi-simple Lie algebra and the semi-direct product Lie algebra induced by the adjoint action of K on $\mathfrak{p}$. This means that the affine-quadratic problem on a semi-simple Lie group G then admits analogous formulation on the semi-direct product $G_s = \mathfrak{p} \rtimes K$. Hence, the semi-direct affine-quadratic problem is always present behind every semi-simple affine problem. We refer to this semi-direct affine problem as *the shadow problem.* We then show that every affine system is controllable whenever A is a regular element in $\mathfrak{p}$. This fact implies that the corresponding affine-quadratic problem is well posed for any positive-definite quadratic form on $\mathfrak{k}$.

On semi-simple Lie groups G with K compact and with a finite center, the Killing form is negative-definite on $\mathfrak{k}$ and can be used to define an Ad_K invariant, positive-definite bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{k}$. The corresponding optimal control system is Ad_K-invariant and hence can be regarded as the canonical affine-quadratic problem on G. It is then natural to consider the departures from the canonical case defined by a quadratic form $\langle Q(u), v \rangle$ for some linear transformation Q on $\mathfrak{k}$ which is positive-definite relative to $\langle\ ,\ \rangle$.

Any such affine-quadratic problem induces a left-invariant affine Hamiltonian

$$H = \frac{1}{2}\langle Q^{-1}(L_{\mathfrak{k}}), L_{\mathfrak{k}} \rangle + \langle A, L_{\mathfrak{p}} \rangle \tag{I.5}$$

on the Lie algebra $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ obtained by the Maximum Principle, where $L_{\mathfrak{k}}$ and $L_{\mathfrak{p}}$ denote the projections of an element $L \in \mathfrak{g}$ on the factors $\mathfrak{k}$ and $\mathfrak{p}$. The Hamiltonians which admit a spectral representation of the form

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$$\frac{dL_\lambda}{dt} = [M_\lambda, L_\lambda] \text{ with} \\ M_\lambda = Q^{-1}(L_\mathfrak{k}) - \lambda A, \text{ and } L_\lambda = -L_\mathfrak{p} + \lambda L_\mathfrak{h} + (\lambda^2 - s)B \tag{I.6}$$

for some matrix B, are called *isospectral*. In this notation s is a parameter, equal to zero in the semi-simple case and equal to one in the semi-direct case.

The spectral invariants of $L_\lambda = L_\mathfrak{p} - \lambda L_\mathfrak{k} + (\lambda^2 - s)B$ are constants of motion and are in involution with each other relative to the Poisson structure induced by either the semi-simple Lie algebra $\mathfrak{g}$ or by the semi-direct product $\mathfrak{g}_s = \mathfrak{p} \ltimes \mathfrak{k}$ (see [Rm], also [Bv; RT]).

We show that an affine Hamiltonian H is isospectral if and only

$$[Q^{-1}(L_\mathfrak{k}), A] = [L_\mathfrak{k}, B] \tag{I.7}$$

for some matrix $B \in \mathfrak{p}$ that commutes with A. In the isospectral case every solution of the homogeneous part

$$\frac{dL_\mathfrak{k}}{dt} = [Q^{-1}(L_\mathfrak{k}), L_\mathfrak{k}] \tag{I.8}$$

is the projection of a solution $L_\mathfrak{p} = sB$ of the affine Hamiltonian system (I.6) and hence admits a spectral representation

$$\frac{dL_\mathfrak{k}}{dt} = [Q^{-1}(L_\mathfrak{k}) - \lambda A, L_\mathfrak{k} - \lambda B]. \tag{I.9}$$

The above shows that the fundamental results of A. T. Fomenko and V. V. Trofimov [Fa] based on Manakov's seminal work on the n-dimensional Euler's top are subordinate to the isospectral properties of affine Hamitonian systems on $\mathfrak{g}$, in the sense that the spectral invariants of $L_\mathfrak{k} - \lambda B$ are always in involution with a larger family of functions generated by the spectral invariants of $L_\lambda = -L_\mathfrak{p} + \lambda L_\mathfrak{h} + (\lambda^2 - s)B$ on $\mathfrak{g}$.

The spectral invariants of L_λ belong to a larger family of functions on the dual of the Lie algebra whose members are in involution with each other, and are sufficiently numerous to guarantee integrability in the sense of Liouville on each coadjoint orbit in $\mathfrak{g}^*$ [Bv].

We then show that the cotangent bundles of space forms, as well as the cotangent bundles of oriented Stiefel and oriented Grassmannian manifolds can be realized as the coadjoint orbits in the space of matrices having zero trace, in which case the restriction of isospectral Hamiltonians to these orbits results in integrable Hamiltonians on the underlying manifolds. In particular, we show that the restriction of the canonical affine Hamiltonian to the

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cotangent bundles of non-Euclidean space forms (spheres and hyperboloids) is given by

$$H = \frac{1}{2}||x||_\epsilon^2||y||_\epsilon^2 - \frac{1}{2}(Ax,x)_\epsilon, \epsilon = \pm 1. \tag{I.10}$$

This Hamiltonian governs the motion of a particle on the space form under a quadratic potential $V = \frac{1}{2}(Ax,x)_\epsilon$. We then show that all of these mechanical systems are completely integrable by computing the integrals of motion generated by the spectral invariants of the matrix L_λ. These integrals of motion coincide with the ones presented by J. Moser in [Ms2] in the case of C. Newmann's system on the sphere.

Remarkably, the degenerate case $A = 0$ provides a natural explanation for the enigmatic discovery of V.A. Fock that the solutions of Kepler's problem move along the geodesics of the space forms [Fk; Ms1; O1; O2]. We show that the stereographic projections from the sphere, respectively the hyperboloid, can be extended to the entire coadjoint orbit in such a way that the extended map is a symplectomorphism from the coadjoint orbit onto the cotangent bundle of $\mathbb{R}^n/\{0\}$ such that $H = \frac{1}{2}(x,x)_\epsilon(y,y)_\epsilon$ is mapped onto $E = \frac{1}{2}||p||^2 - \frac{1}{||q||}$ and the energy level $H = \frac{\epsilon}{2h^2}$ is mapped onto the energy level $E = -\frac{1}{2}\epsilon h^2$. Therefore $E < 0$ in the spherical case and $E > 0$ in the hyperbolic case. The Euclidean case $E = 0$ is obtained by a limiting argument when ϵ is regarded as a continuous parameter which tends to zero. This correspondence also identifies the angular momentum and the Runge–Lenz vector associated with the problem of Kepler with the moment map associated with the Hamiltonian H.

The chapter on the matrices in $sl_{n+1}(R)$ also includes a discussion of a left-invariant geodesic problem on the group of upper triangular matrices that is relevant for the solutions of a Toda lattice system. Our exposition then turns to Jacobi's geodesic problem on the ellipsoid and the origins of its integrals of motion. We show that there is a surprising and beautiful connection between this classical problem and isospectral affine Hamiltonians on $sl_n(R)$ that sheds much light on the symmetries that account for the integrals of motion. The path is somewhat indirect: rather than starting with Jacobi's problem on the ellipsoid $x \cdot D^{-1}x = 1$, we begin instead with a geodesic problem on the sphere in which the length is given by the elliptic metric $\int_0^T \sqrt{(Dx(t) \cdot x(t)}\, dt$. It turns out that the Hamiltonian system corresponding to the elliptic problem on the sphere is symplectomorphic to the Hamiltonian system associated with the geodesic problem of Jacobi on the ellipsoid, but in contrast to Jacobi's problem, the Hamiltonian system on the sphere can be represented as a coadjoint orbit. It turns out that the Hamiltonian system associated with the elliptic problem on

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the sphere is equal to the restriction of an isospectral affine Hamiltonian system H on $sl_{n+1}(R)$, and hence inherits the integrals of motion from the spectral matrix L_λ. In fact, the Hamiltonian is given by

$$H = \frac{1}{2}\langle D^{-1}L_{\mathfrak{k}}D^{-1}, L_{\mathfrak{k}}\rangle + \langle D^{-1}, L_{\mathfrak{p}}, \rangle \tag{I.11}$$

and its spectral matrix by $L_\lambda = L_{\mathfrak{p}} - \lambda L_{\mathfrak{k}} + (\lambda^2 - s)D$. This observation reveals that the mechanical problem of Newmann and the elliptic problem on the sphere share the same integrals of motion. This discovery implies not only that all three problems – the mechanical problem of Newmann, Jacobi's problem on the ellipsoid and the elliptic problem on the sphere – are integrable, but it also identifies the symmetries that account for their integrals of motion. These findings validate Moser's speculation that the symmetries that account for these integrals of motion are hidden in the Lie algebra $sl_{n+1}(R)$ [Ms3].

The material then shifts to the rigid body and the seminal work of S. V. Manakov mentioned earlier. We interpret Manakov's integrability results in the realm of isospectral affine Hamiltonians, and provide natural explanations for the integrability of the equations of motion for a rigid body in the presence of a quadratic Newtonian field (originally discovered by O. Bogoyavlensky in 1984 [Bg1].

We then consider the Hamiltonians associated with the affine-quadratic problems on the isometry groups $SE_3(R)$, $SO_4(R)$ and $SO(1,3)$. These Hamiltonians contain six parameters: three induced by the left-invariant metric and another three corresponding to the coordinates of the drift vector. The drift vector reflects how the tangent of the curve is related to the orthonormal frame along the curve. To make parallels with a heavy top, we associate the metric parameters with the principal moments of inertia and the coordinates of the drift vector with the coordinates of the center of gravity. Then we show that these Hamiltonians are integrable precisely under the same conditions as the heavy tops, with exactly three integrable cases analogous to the top of Euler, top of Lagrange and the top of Kowalewski.

The fact that the Lie algebras $so_4(R)$ and $so(1,3)$ are real forms for the complex Lie algebra $so_4(\mathbb{C})$ suggests that the Hamiltonian equations associated with Kirchhoff's problem should be complexified and studied on $so_4(\mathbb{C})$ rather than on the real Lie algebras. This observation seems particularly relevant for the Kowalewski case. We show that the Hamiltonian system that corresponds to her case admits four holomorphic integrals of motion, one of which is of the form

$$I_4 = \left(\frac{1}{2\lambda}z_1^2 - bw_1 + s\frac{\lambda}{2}b^2\right)\left(\frac{1}{2\lambda}z_2^2 - \bar{b}w_2 + s\frac{\lambda}{2}\bar{b}^2\right),$$

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where $s = 0$ corresponds to the semi-direct case and $s = 1$ to the semi-simple case. For $s = 0$ and $\lambda = 1$ this integral of motion coincides with the one obtained by S. Kowalewski in her famous paper of 1889 [Kw]. The passage to complex Lie algebras validates Kowalewska's mysterious use of complex variables and also improves the integration procedure reported in [JA].

Our treatment of the above Hamiltonians reveals ubiquitous presence of elastic curves in these Hamitonians. Elastic curves are the projections of extremal curves associated with the functional $\frac{1}{2}\int_0^T \kappa^2(s)\,ds$. In Chapter 16 we consider this problem in its own right as the curvature problem. Parallel to the curvature problem we also consider the problem of finding a curve of shortest length among the curves that satisfy fixed tangential directions at their ends and whose curvature is bounded by a given constant c. This problem is referred to as the Dubins–Delauney problem. Our interest in Delauney–Dubins problem is inspired by a remarkable paper of L. Dubins of 1957 [Db] in which he showed that optimal solutions exist in the class of continuosly differentiable curves having Lebesgue integrable second derivatives, and characterized optimal solutions in the plane as the concatenations of arcs of circles and straight line segments with the number of switchings from one arc to another equal to at most two.

We will show that the solutions of n-dimensional Dubins' problem on space forms are essentially three dimensional and are characterized by two integrals of motion I_1 and I_2. Dubins' planar solutions persist on the level $I_2 = 0$, while on $I_2 \neq 0$ the solutions are given by elliptic functions obtained exactly as in the paper of J. von Schwarz of 1934 in her treatment of the problem of Delaunay [VS]. Our solutions also clarify Caratheodory's fundamental formula for the problem of Delauney at the end of his book on the calculus of variations. [Cr, p. 378].

This chapter also includes a derivation of the Hamiltonian equation associated with the curvature problem on a general symmetric space G/K corresponding to the Riemannian symmetric pair (G, K). The corresponding formulas show clear dependence of this problem on the Riemannian curvature of the underlying space. We then recover the known integrability results on the space forms explained earlier in the book, and show the connections with rolling sphere problems discovered in [JZ].

The book ends with with a brief treatment of infinite-dimensional Hamiltonian systems and their relevance for the solutions of the non-linear Schroedinger equation, the Korteveg–de Vries equation and Heisenberg's magnetic equation. This material is largely inspired by another spectacular property of the elastic curves – they appear as the soliton solutions in the non-linear Schroedinger equation. We will be able to demonstrate this fact by

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introducing a symplectic structure on an infinite-dimensional Fréchet manifold of framed curves of fixed length over a three-dimensional space form. We will then use the symplectic form to identify some partial differential equations of mathematical physics with the Hamiltonian flows generated by the functionals defined by the geometric invariants of the underlying curves, such as the curvature and the torsion functionals.

Keeping in mind the reader who may not be familiar with all aspects of this theory we have made every effort to keep the exposition self-contained and integrated in a way that minimizes the gap between different fields. Unavoidably, some aspects of the theory have to be taken for granted such as the basic knowledge of manifolds and differential equations.

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