# CONTENTS

Preface  

1 The Orbits of One-Dimensional Maps  
   1.1 Iteration of Functions and Examples of Dynamical Systems  
   1.2 Newton’s Method and Fixed Points  
   1.3 Graphical Iteration  
   1.4 The Stability of Fixed Points  
   1.5 Non-Hyperbolic Fixed Points  

2 Bifurcations and the Logistic Family  
   2.1 The Basin of Attraction  
   2.2 The Logistic Family  
   2.3 Periodic Points  
   2.4 Periodic Points of the Logistic Map  
   2.5 The Period Doubling Route to Chaos  
   2.6 The Bifurcation Diagram and 3-Cycles of the Logistic Map  
   2.7 The Tent Family $T_\mu$  
   2.8 The 2-Cycles and 3-Cycles of the Tent Family  

3 Sharkovsky’s Theorem  
   3.1 Period 3 Implies Chaos  
   3.2 Converse of Sharkovsky’s Theorem  

4 Dynamics on Metric Spaces  
   4.1 Basic Properties of Metric Spaces  
   4.2 Dense Sets  
   4.3 Functions between Metric Spaces  
   4.4 Diffeomorphisms of $\mathbb{R}$  

5 Countability, Sets of Measure Zero and the Cantor Set  
   5.1 Countability and Sets of Measure Zero  
   5.2 The Cantor Set  

Preface xi  

1 The Orbits of One-Dimensional Maps 1  
   1.1 Iteration of Functions and Examples of Dynamical Systems 1  
   1.2 Newton’s Method and Fixed Points 10  
   1.3 Graphical Iteration 17  
   1.4 The Stability of Fixed Points 20  
   1.5 Non-Hyperbolic Fixed Points 28  

2 Bifurcations and the Logistic Family 38  
   2.1 The Basin of Attraction 38  
   2.2 The Logistic Family 40  
   2.3 Periodic Points 45  
   2.4 Periodic Points of the Logistic Map 52  
   2.5 The Period Doubling Route to Chaos 54  
   2.6 The Bifurcation Diagram and 3-Cycles of the Logistic Map 56  
   2.7 The Tent Family $T_\mu$ 64  
   2.8 The 2-Cycles and 3-Cycles of the Tent Family 66  

3 Sharkovsky’s Theorem 70  
   3.1 Period 3 Implies Chaos 70  
   3.2 Converse of Sharkovsky’s Theorem 74  

4 Dynamics on Metric Spaces 79  
   4.1 Basic Properties of Metric Spaces 79  
   4.2 Dense Sets 83  
   4.3 Functions between Metric Spaces 88  
   4.4 Diffeomorphisms of $\mathbb{R}$ 94  

5 Countability, Sets of Measure Zero and the Cantor Set 100  
   5.1 Countability and Sets of Measure Zero 100  
   5.2 The Cantor Set 105
# Contents

5.3 Ternary Expansions and the Cantor Set 108  
5.4 The Tent Map for $\mu = 3$ 111  
5.5 A Cantor Set Arising from the Logistic Map $L_\mu$, $\mu > 4$ 114  

6 Devaney’s Definition of Chaos 116  
6.1 The Doubling Map and the Angle Doubling Map 117  
6.2 Transitivity 120  
6.3 Sensitive Dependence on Initial Conditions 121  
6.4 The Definition of Chaos 122  
6.5 Symbolic Dynamics and the Shift Map 126  
6.6 For Continuous Maps, Sensitive Dependence is Implied by Transitivity and Dense Period Points 130  

7 Conjugacy of Dynamical Systems 134  
7.1 Conjugate Maps 134  
7.2 Properties of Conjugate Maps and Chaos Through Conjugacy 137  
7.3 Linear Conjugacy 142  

8 Singer’s Theorem 146  
8.1 The Schwarzian Derivative Revisited 146  
8.2 Singer’s Theorem 150  

9 Conjugacy, Fundamental Domains and the Tent Family 154  
9.1 Conjugacy and Fundamental Domains 154  
9.2 Conjugacy, the Tent Map and Periodic Points of the Tent Family 158  

10 Fractals 166  
10.1 Examples of Fractals 166  
10.2 An Intuitive Introduction to the Idea of Fractal Dimension 168  
10.3 Box Counting Dimension 170  
10.4 The Mathematical Theory of Fractals 174  
10.5 The Contraction Mapping Theorem and Self-Similar Sets 176  

11 Newton’s Method for Real Quadratics and Cubics 184  
11.1 Binary Representation of Real Numbers 184  
11.2 Newton’s Method for Real Quadratic Polynomials 186  
11.3 Newton’s Method for Real Cubic Polynomials 188  
11.4 The Cubic Polynomials $f_\lambda(x) = (x + 2)(x^2 + c)$ 190  

12 Coppel’s Theorem and a Proof of Sharkovsky’s Theorem 195  
12.1 Coppel’s Theorem 195
12.2 The Proof of Sharkovsky’s Theorem  199
12.3 The Completion of the Proof of Sharkovsky’s Theorem  203

13 Real Linear Transformations, the Hénon Map and Hyperbolic Toral Automorphisms  209
  13.1 Linear Transformations  209
  13.2 The Hénon Map  216
  13.3 Circle Maps Induced by Linear Transformations on $\mathbb{R}$  220
  13.4 Endomorphisms of the Torus  221
  13.5 Hyperbolic Toral Automorphisms  224

14 Elementary Complex Dynamics  228
  14.1 The Complex Numbers  228
  14.2 Analytic Functions in the Complex Plane  229
  14.3 The Dynamics of Polynomials and the Riemann Sphere  234
  14.4 The Julia Set  239
  14.5 The Mandelbrot Set $M$  249
  14.6 Newton’s Method in the Complex Plane for Quadratics and Cubics  256
  14.7 Important Complex Functions  263

15 Examples of Substitutions  271
  15.1 One-Dimensional Substitutions and the Thue–Morse Substitution  271
  15.2 The Toeplitz Substitution  278
  15.3 The Rudin–Shapiro Sequence  281
  15.4 Paperfolding Sequences  284

16 Fractals Arising from Substitutions  290
  16.1 A Connection between the Morse Substitution and the Koch Curve  290
  16.2 Dragon Curves  293
  16.3 Fractals Arising from Two-Dimensional Substitutions  294
  16.4 The Rauzy Fractal  300

17 Compactness in Metric Spaces and an Introduction to Topological Dynamics  311
  17.1 Compactness in Metric Spaces  311
  17.2 Continuous Functions on Compact Metric Spaces  316
  17.3 The Contraction Mapping Theorem for Compact Metric Spaces  318
## Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.4</td>
<td>Basic Topological Dynamics</td>
<td>319</td>
</tr>
<tr>
<td>17.5</td>
<td>Topological Mixing and Exactness</td>
<td>328</td>
</tr>
<tr>
<td>18</td>
<td>Substitution Dynamical Systems</td>
<td>336</td>
</tr>
<tr>
<td>18.1</td>
<td>Sequence Spaces</td>
<td>336</td>
</tr>
<tr>
<td>18.2</td>
<td>Languages</td>
<td>342</td>
</tr>
<tr>
<td>18.3</td>
<td>Dynamical Systems Arising from Sequences</td>
<td>344</td>
</tr>
<tr>
<td>18.4</td>
<td>Substitution Dynamics</td>
<td>350</td>
</tr>
<tr>
<td>19</td>
<td>Sturmian Sequences and Irrational Rotations</td>
<td>355</td>
</tr>
<tr>
<td>19.1</td>
<td>Sturmian Sequences</td>
<td>355</td>
</tr>
<tr>
<td>19.2</td>
<td>Sequences Arising from Irrational Rotations</td>
<td>359</td>
</tr>
<tr>
<td>19.3</td>
<td>Cutting Sequences</td>
<td>364</td>
</tr>
<tr>
<td>19.4</td>
<td>Sequences Arising from Irrational Rotations are Sturmian</td>
<td>367</td>
</tr>
<tr>
<td>19.5</td>
<td>Semi-Topological Conjugacy between ([0, 1), T_{α}) and (O(τ), σ)</td>
<td>371</td>
</tr>
<tr>
<td>19.6</td>
<td>The Three Distance Theorem</td>
<td>374</td>
</tr>
<tr>
<td>20</td>
<td>The Multiple Recurrence Theorem of Furstenberg and Weiss</td>
<td>379</td>
</tr>
<tr>
<td>20.1</td>
<td>van der Waerden’s Theorem</td>
<td>379</td>
</tr>
<tr>
<td>Appendix A</td>
<td>Theorems from Calculus</td>
<td>383</td>
</tr>
<tr>
<td>Appendix B</td>
<td>The Baire Category Theorem</td>
<td>385</td>
</tr>
<tr>
<td>Appendix C</td>
<td>The Complex Numbers</td>
<td>387</td>
</tr>
<tr>
<td>Appendix D</td>
<td>Weyl’s Equidistribution Theorem</td>
<td>389</td>
</tr>
</tbody>
</table>

References 391

Index 398