Chaotic Dynamics

Fractals, Tilings, and Substitutions

This undergraduate textbook is a rigorous mathematical introduction to dynamical systems and an accessible guide for students transitioning from calculus to advanced mathematics. It has many student-friendly features, such as graded exercises that range from straightforward to more difficult with hints, and includes concrete applications of real analysis and metric space theory to dynamical problems. Proofs are complete and carefully explained, and there are plenty of opportunities to practice manipulating algebraic expressions in an applied context of dynamical problems. After presenting a foundation in one-dimensional dynamical systems, in the latter chapters the text introduces students to advanced subjects, such as topological and symbolic dynamics. It includes two-dimensional dynamics, Sharkovsky's theorem and the theory of substitutions, and takes special care in covering Newton's method. The Mathematica code is available online, so that students can see implementations of many dynamical aspects of the text.

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This book is dedicated to my wife Joyce, my son Garth and my daughters Jacqui and Emma, whose love and support has been my greatest encouragement.

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Frontmatter
More Information

CONTENTS

PREFACE

Many of the most recent International Congresses of Mathematics have awarded Fields Medals to researchers in chaotic dynamics and related fields, indicating the importance of these areas. Dynamics has blossomed in the past 50 years, making it useful as a tool for demonstrating techniques to mathematics majors and for developing their general mathematical maturity. It is my hope that this book will provide interested students with an introduction to one-dimensional dynamical systems, giving them the tools necessary to succeed in more advanced courses on this topic. The early chapters of this book can be used as a stepping stone from the non-rigorous courses of freshman calculus to the more advanced topics of real analysis and topology.

Towson University is a liberal arts college and is part of the University of Maryland System. In my first years of teaching a course on dynamical systems, I based my lectures on the material of some of the existing textbooks which were then currently available, such as [**122**], [**41**], [**65**] and [**32**]. Each semester, I found myself changing the course content and exercises (frequently to meet the needs of my students). This led to the production of my own lecture notes (these notes owe a debt to the above mentioned books).

The content of this text arises primarily from lecture notes that I created over many years of teaching senior seminar-type courses to final year students at Towson University, and also from courses in the Towson University Applied Mathematics Graduate Program and the Graduate Program in Mathematics Education. In the senior seminar course, students were taught the basics of one-dimensional dynamics and were required to present a project at completion of the course. The later chapters of this book include many topics from these projects, for example, Sharkovsky's Theorem, as well as topics resulting from the independent study of some of my Master's students. With students in the Master's Program, I was generally able to move quickly through the earlier material and spend more time on advanced topics, the choice of which changed from semester to semester.

Students and instructors may find the following information useful. The first two chapters are an introduction to the theory of fixed points and periodic points, describing the behavior of maps under iteration. Newton's method, which is an important theme throughout the text, and elementary bifurcation theory are discussed. These are reinforced with concrete examples and numerous exercises. Chapter 3 discusses Sharkovsky's Theorem with a proof of the special case of maps having points of period three (the Li–Yorke Theorem). The full proof of Sharkovsky's Theorem is left until Chapter 12. Chapters 4 and 5 lead the student to metric spaces, generalizing results that appear in Chapters 1 and 2 and including important examples such as the Cantor set and the shift map. In Chapters 6 and 7 the notions of chaos and conjugacy are introduced. Chapters 8 through 14 continue with a study of conjugacy, the Schwarzian derivative, Newton's method, complex dynamics, Sharkovsky's Theorem and some two-dimensional dynamics. The latter third of the book is devoted to topological dynamics on compact spaces and an introduction to substitution dynamics. Throughout this book, I aim to develop the theory in a mathematically rigorous manner. The first 14 chapters (possibly omitting Chapters 11 and 12) cover fairly standard topics in (mostly) one-dimensional dynamics, and should be accessible to upper level undergraduate students. The requirements from real analysis and topology (metric spaces) are developed as the material progresses. The text reinforces some of the theoretical results that students have encountered in calculus, such as the Intermediate Value and Mean Value Theorems. A subject such as chaotic dynamics requires a certain amount of mathematical sophistication. It is certainly an advantage for students to have a background in real analysis prior to taking the course, but it is not essential. I feel that this text allows students who have completed a freshman calculus sequence to be successful in a first course in dynamical systems, if this text is followed. Students who do have a previous background in real analysis will certainly see the importance of real analysis and basic topology in mathematics. I believe that this is not frequently apparent to students in undergraduate studies.

Chapters 15 and 16 give an introduction to the theory of substitutions via examples, and we show how these can give rise to certain types of fractals and tilings. Subsequent chapters develop the rigorous mathematical theory of substitutions and Sturmian sequences. In order to give a rigorous account of symbolic dynamical systems in Chapters 18 and 19, Chapter 17 is devoted to topological dynamical systems, developing the necessary theory and also introducing concepts related to the material that has preceded it. The final chapter touches on the Multiple Recurrence Theorem of Furstenburg and Weiss (topological version).

This is a mathematical text and I have not focused on applications. Most expositions of one-dimensional dynamical systems are non-rigorous at the undergraduate level or assume a level of sophistication above that of the upper level undergraduate or beginning graduate student. I believe that this book is suitable for a first course in dynamical systems at the junior or senior level (or even at the sophomore level at some schools). It can also be used in a seminar course on substitution dynamics following a basic dynamics course, or as a supplement for projects in a standard course. A second course can be given to advanced undergraduates, either through independent study or as a course utilizing the material of Chapters 9, 12 and 15–20 (also 14 if not included earlier).

Our study of substitutions is combinatorial and topological, avoiding any measure theory. Consequently, we do not touch on some important topics of current interest such as the spectral properties of substitutions (see [**104**]). It is hoped that our choice of topics will encourage the readers to continue their studies in some aspect of dynamical systems, such as ergodic theory, topological dynamics, differentiable dynamics, symbolic dynamics, tiling dynamical systems or complex dynamics. We have included much more material than can be covered in a one-semester course. A one-semester course could consist of Chapters 1 through 14, possibly omitting Chapters 11 and 12. Depending on time constraints, and the level of the student, some or all of Chapters 9 and 13 might also be omitted. If necessary, various topics may be given to the students to read on their own or omitted altogether. These include Section 2.8 concerning the tent map, Section 4.4 concerning diffeomorphisms on $\mathbb{R}$ and Section 6.6 giving the dependency of certain conditions in the definition of chaos. Section 14.7 is mostly of historical interest, giving the original proof due to Schröeder of what is often called Cayley's Theorem, and may easily be omitted.

Each chapter and many of the sections are accompanied by exercises that aim to lead the student to a better understanding of the material. An asterisk * is used to indicate more difficult problems. Much of the material in this text owes a debt to quite recent publications in the field of dynamical systems appearing in journals such as the *American Mathematical Monthly*, the *College Mathematics Journal*, *Mathematical Intelligencer* and *Mathematics Magazine*, and it would be very valuable for students to read some of this original material. Various Internet resources have been used, such as Wolfram's MathWorld, some of these without citation because of difficulties in identifying the author. All of the figures in the text were created using LaTeX or the computer algebra system Mathematica. The text was typeset with LaTeX. Computer algebra systems are indispensable tools for studying all aspects of this subject. We have sometimes used a computer algebra system to simplify complicated algebraic manipulations but have generally avoided its use when possible. In teaching this course, I have used Mathematica to illustrate the concepts. A supplement to this course containing the Mathematica code used can be downloaded at: www.cambridge.org/goodson.

The lecture notes on which this book is based benefited tremendously from being read by the students in my class. In particular, I would like to recognize three of my Master's students: Nirmal Malapaka, Christopher Jones and

Albandary Alshahrami, all of whom produced Mathematica files that have enabled me to include many of the figures in this text.

I would like to thank the teachers who first introduced me to ergodic theory, and colleagues who were instrumental for arousing my interest in dynamical systems. These include William Parry, Peter Walters and Rufus Bowen at the University of Warwick, UK, Dan Newton at the University of Sussex, UK, Michael Sears and Harvey Keynes at the University of the Witwatersrand, Johannesburg and Dan Rudolph at the University of Maryland.

I am indebted to Towson University and my colleagues there, Angel Kumchev and Houshang Sohrab, for reading parts of this text and for their valuable input. I am grateful to the early reviewers (before this text was fully written), for their insight and excellent suggestions which have made this a better book. I would like to thank my wife Joyce, who with a keen mathematical eye read this text. This project would have been impossible without her support, invaluable help and understanding. Finally, I would like to thank Kaitlin Leach at Cambridge University Press for her interest in this book, and for all the help from people at the Press in moving this book along. Special thanks to the copy-editor Jon Billam, for his careful scrutiny of the text prior to publication.

I am solely responsibility for any errors that may have occurred, and I welcome any comments from the readers of this book.

Chapter Dependency