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1 The Orbits of One-Dimensional Maps

In this chapter we introduce one-dimensional dynamical systems and analyze some elementary examples. A study of the iteration in Newton's method leads naturally to the notion of attracting fixed points for dynamical systems. Newton's method is emphasized throughout as an important motivation for the study of dynamical systems. The chapter concludes with various criteria for establishing the stability of the fixed points of a dynamical system.

1.1 Iteration of Functions and Examples of Dynamical Systems

Chaotic dynamical systems has its origins in Henri Poincaré's memoir on celestial mechanics and the three-body problem (1890s). Poincaré's memoir arose from his entry in a competition celebrating the 60th birthday of King Oscar of Sweden. His manuscript concerned the stability of the solar system and the question of how three bodies, with mutual gravitational interaction, behave. This was a problem that had been solved for two bodies by Isaac Newton. Although Poincaré was not able to determine exact solutions to the three-body problem, his study of the long term behavior of such dynamical systems resulted in a prize winning manuscript. In particular, he claimed that the solutions to the three-body problem (restricted to the plane) are stable, so that a solar system such as ours would continue orbiting more or less as it does, forever. After the competition, and when his manuscript was ready for publication, he noticed it contained a deep error which showed that instability may arise in the solutions. In correcting the error, Poincaré discovered chaos and his memoir became one of the most influential scientific publications of the past century [**10**]. Aspects of dynamical systems were already evident in the study of iteration in Newton's method for approximating the zeros of functions. The work of Cayley and Schroeder concerning Newton's method in the complex domain appeared during the 1880s, and interest in this new field of complex dynamics continued in the early 1900s with the work of Fatou and Julia. Their work lay dormant until the invention of the electronic computer. In the 1960s the subject exploded into life with the work of Sharkovsky and Li and Yorke on one-dimensional dynamics, and with that of Kolmogorov, Smale, Anosov and others on differentiable dynamics and ergodic theory. The advent of computer graphics allowed for the resurgence of complex dynamics and the depiction of fractals (Devaney and Mandelbrot).

This book is mainly concerned with one-dimensional dynamical systems for real and complex mappings and their connections with fractal geometry. We also treat certain symbolic dynamical systems in detail; in particular we look at substitution dynamical systems and the fractals they generate.

Dynamical systems involves the study of how things change over time. Examples include the growth of populations, the change in the weather, radioactive decay, mixing of liquids such as the ocean currents, motion of the planets, the interest in a bank account. Some of these dynamical systems are well behaved and predictable; for example, if you know how much money you have in the bank today, it should be possible to calculate how much you will have next month (based on how much you deposit, the interest rate etc.). However, some dynamical systems are inherently unpredictable and so are called chaotic. An example of this is weather forecasting, which is generally unreliable in predicting the weather beyond the next three or four days. Intuition tells us that chaotic behavior will happen provided we have some degree of randomness in the system. However, chaos can happen even when the dynamical system is deterministic, that is, its future behavior is completely determined by its initial conditions. To quote Edward Lorenz, who was the first to realize that deterministic chaos is present in weather forecasting: chaos occurs "when the present determines the future, but the approximate present does not approximately determine the future." In theory, if we could measure exactly the weather at some instant in time at every point in the Earth's atmosphere, we could predict how it will behave in the future. But because we can only approximately measure the weather (wind speed and direction, temperature etc.), the future weather is unpredictable.

Throughout we use $\mathbb{R}$ to denote the set of real numbers, $\mathbb{Z} = \{\ldots, -1, 0, 1, 2, 3, \ldots\}$ is the set of integers, $\mathbb{N} = \{0, 1, 2, \ldots\}$ are the natural numbers, $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ are the positive integers and $\mathbb{Q}$ is the set of rational numbers.

Dynamical systems with continuously varying time, which are called *flows*, arise from the solutions to differential equations. In this text, we will study *discrete dynamical systems*, arising from discrete changes in time. For example, we might model a population by measuring it daily. Suppose that x_n is the number of members of a population on day n, where x_0 is the initial population. We look for a function $f : \mathbb{R} \to \mathbb{R}$, for which

$$x_1 = f(x_0), \quad x_2 = f(x_1), \quad \text{and generally} \quad x_n = f(x_{n-1}), \quad n = 1, 2, \ldots$$

This leads to the iteration of functions in the following way:

Definition 1.1.1 Given $x_0 \in \mathbb{R}$, the *orbit* of x_0 under f is the set

$$O(x_0) = \{x_0, f(x_0), f^2(x_0), \ldots\},$$

where $f^2(x_0) = f(f(x_0)), f^3(x_0) = f(f^2(x_0))$, continuing indefinitely, so that

$$f^n(x) = f \circ f \circ f \circ \cdots \circ f(x) \quad (n\text{-times composition}).$$

For each $n \in \mathbb{N}$, set $x_n = f^n(x_0)$; then $x_1 = f(x_0)$, $x_2 = f^2(x_0)$, and in general

$$x_{n+1} = f^{n+1}(x_0) = f(f^n(x_0)) = f(x_n).$$

More generally, f may be defined on some subinterval I of $\mathbb{R}$, but in order for the iterates of $x \in I$ under f to be defined, we need the *range* of f to be contained in I, so $f : I \to I$ (both the *domain* and the *codomain* of f are the same set).

Thus we are studying the iterations of *one-dimensional maps* (as opposed to higher dimensional maps $f : \mathbb{R}^n \to \mathbb{R}^n$, $n > 1$, some of which will be considered in Chapter 13).

Definition 1.1.2 A (one-dimensional) *dynamical system* is a pair (I, f), where f is a function $f : I \to I$ and I is a subset of $\mathbb{R}$.

Almost always, I will be a subinterval of $\mathbb{R}$, which includes the possibility that $I = \mathbb{R}$.

Often we will talk about the dynamical system $f : I \to I$, or just f when the domain is clear. Usually, f is assumed to be a continuous function, but we occasionally relax this requirement. For example, $f : [0,1] \to [0,1], f(x) = x^2$ and $g : [0,1] \to [0,1]$, $g(x) = 2x$ if $0 \leq x < 1/2$ and $g(x) = 2x - 1$ if $1/2 \leq x \leq 1$ are dynamical systems (the latter is not continuous), but $h : [0,2] \to [0,4]$, $h(x) = x^2$ is not a dynamical system, since the domain and codomain are different.

Given a dynamical system f, equations of the form $x_{n+1} = f(x_n)$ are examples of *difference equations*. These arise from one-dimensional dynamical systems. For example, x_n may represent the number of bacteria in a culture after n hours, or the mass of radioactive material remaining after n days of an experiment. There is an obvious correspondence between one-dimensional maps and these difference equations. For example, a difference

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equation commonly used for calculating square roots,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right),$$

corresponds to the function $f(x) = \frac{1}{2}(x + \frac{2}{x})$. If we start by setting $x_0 = 2$ (or in fact any real number), and then find $x_1, x_2, \ldots$ etc., we get a sequence which rapidly approaches $\sqrt{2}$ (see p. 9 of Sternberg [**122**]). One of the issues we examine in this chapter is how this difference equation arises and its usefulness in calculating square roots.

Examples 1.1.3 Dynamical Systems

1. **The Trigonometric Functions** Consider the iterations of the trigonometric function $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin x$. Select $x_0 \in \mathbb{R}$ at random, e.g., $x_0 = 2$ and set $x_{n+1} = \sin x_n,\ n = 0, 1, 2, \ldots$ What happens to x_n as n increases? One way to investigate this type of dynamical system is to use a graphing utility: enter **Sin**(2), followed by **ENTER**, and then **Sin**(**ANSWER**), and then continue this process. You will need to do this many times to get a good idea of what is happening. It may be easier to use a computer algebra system to carry out the computations.

Now replace the sine function with the cosine function and repeat the process. How do we explain what appears to be happening in each case? These are questions that will be answered in this chapter.

2. **Linear Maps** These are possibly the simplest dynamical systems for modeling population growth and also the easiest to deal with from a dynamical point of view, since we can obtain a clear description of their long term behavior. Every linear map $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = ax$ for some $a \in \mathbb{R}$. Suppose that x_n = size of a population at time n, with the property

$$x_{n+1} = ax_n,$$

for some constant $a > 0$. This is an example of a *linear model* for the growth of the population.

If the initial population is $x_0 > 0$, then $x_1 = ax_0$, $x_2 = ax_1 = a^2x_0$, and in general $x_n = a^n x_0$ for $n = 0, 1, 2, \ldots$ This is the exact solution (or *closed form solution*) to the difference equation $x_{n+1} = ax_n$. Clearly $f(x) = ax$ is the corresponding dynamical system. We can use the solution to determine the long term behavior of the population:

The sequence (x_n) is very well behaved since:

(i) if $a > 1$, then $x_n \to \infty$ as $n \to \infty$,
(ii) if $0 < a < 1$ then $x_n \to 0$ as $n \to \infty$ (i.e., the population becomes extinct),
(iii) if $a = 1$, then the population remains unchanged.

3. **Affine maps** These are functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$ $(a \neq 0)$, for constants a and b. Consider the iterates of such maps:

$$f^2(x) = f(ax+b) = a(ax+b) + b = a^2x + ab + b,$$
$$f^3(x) = a^3x + a^2b + ab + b,$$
$$f^4(x) = a^4x + a^3b + a^2b + ab + b,$$

and generally

$$f^n(x) = a^nx + a^{n-1}b + a^{n-2}b + \cdots + ab + b.$$

Let $x_0 \in \mathbb{R}$ and set $x_n = f^n(x_0)$; then we have

$$\begin{aligned} x_n &= a^nx_0 + (a^{n-1} + a^{n-2} + \cdots + a + 1)b \\ &= a^nx_0 + b\left(\frac{a^n - 1}{a - 1}\right), \quad \text{if } a \neq 1, \end{aligned}$$

or

$$x_n = \left(x_0 + \frac{b}{a-1}\right)a^n + \frac{b}{1-a}, \quad \text{if } a \neq 1,$$

is the closed form solution. Here we have used the formula for the sum of a finite geometric series:

$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1},$$

when $r \neq 1$. If $a = 1$, the solution is $x_n = x_0 + nb$.

We can use these equations to determine the long term behavior of x_n. We see that:

(i) if $|a| < 1$ then $a^n \to 0$ as $n \to \infty$, so that

$$\lim_{n\to\infty} x_n = \frac{b}{1-a},$$

(ii) if $a > 1$, then $\lim_{n\to\infty} x_n = \infty$ for $b, x_0 > 0$,
(iii) if $a = 1$, then $\lim_{n\to\infty} x_n = \infty$ if $b > 0$.

The limit does not exist if $a \leq -1$ (unless $x_0 + b/(a-1)) = 0$.

1.1.4 Recurrence Relations Many sequences can be defined *recursively* by specifying the first few terms, and then stating a general rule which specifies how to obtain the nth term from the $(n-1)$th term (or other additional terms), and using mathematical induction to see that the sequence is "well defined" for every $n \in \mathbb{N}$. For example, $n! = n$-factorial can be defined in this way by specifying $0! = 1$ and $n! = n(n-1)!$, for $n \in \mathbb{Z}^+$. The *Fibonacci sequence* (F_n) can be defined by setting

$$F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n, \quad \text{for} \quad n \geq 0,$$

so that $F_2 = 1$, $F_3 = 2$, giving the sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

The orbit of a point $x_0 \in \mathbb{R}$ under a function f is then defined recursively as follows:

$$x_n = f(x_{n-1}), \quad \text{for} \quad n \in \mathbb{Z}^+,$$

with a given starting value x_0. The principle of mathematical induction tells us that x_n is defined for every $n \geq 0$, since it is defined for $n = 0$. Assuming it has been defined for $k = n - 1$ then $x_n = f(x_{n-1})$ defines it for $k = n$.

Ideally, given a recursively defined sequence (x_n), we would like to have a specific formula for x_n in terms of elementary functions (a so called *closed form solution*). This is often very difficult or impossible to achieve. In the case of affine maps and certain logistic maps, however, there is a closed form solution. One can use these ideas to study certain problems, as illustrated in the following examples.

Example 1.1.5 An amount \$$T$ is deposited in your bank account at the end of each month. The interest is r% per month. Find the amount $A(n)$ accumulated at the end of n months (assume $A(0) = T$).

Answer $A(n)$ satisfies the difference equation

$$A(n+1) = A(n) + A(n)r/100 + T, \quad \text{where } A(0) = T,$$

or

$$A(n+1) = A(n)(1 + r/100) + T.$$

Setting $x_0 = T$, $a = 1 + r/100$ and $b = T$ in the formula of Example 1.1.3(3), the solution is

$$\begin{aligned} A(n) &= (1 + r/100)^n T + T\left(\frac{(1 + r/100)^n - 1}{1 + r/100 - 1}\right) \\ &= (1 + r/100)^n T + 100\frac{T}{r}((1 + r/100)^n - 1). \end{aligned}$$

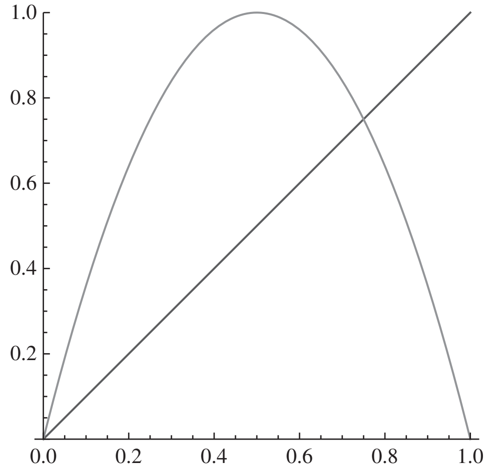
1.1.6 The Logistic Map In the late 1940s, John von Neumann proposed that the map given by $f(x) = 4x(1-x)$ could be used as a pseudo-random number generator. Maps of this type were amongst the first to be studied using electronic computers. Paul Stein and Stanislaw Ulam did an extensive computer study of $f(x)$ and related maps in the early 1950s, but much about these maps remained mysterious.

More generally, maps of the form

$$L_\mu : \mathbb{R} \to \mathbb{R}, \quad L_\mu(x) = \mu x(1-x),$$

were proposed to model a certain type of population growth (see the work of Robert May [**87**]). Here μ is a real parameter which is fixed. Note that if $0 < \mu \leq 4$, then L_μ is a dynamical system of the interval $[0, 1]$, i.e. $L_\mu : [0, 1] \to [0, 1]$. For example, when $\mu = 4$, $L_4(x) = 4x(1-x)$, with $L_4([0, 1]) = [0, 1]$; the graph is given in the figure below. If $\mu > 4$, L_μ is no longer a dynamical system of $[0, 1]$ as $L_\mu([0, 1])$ is not a subset of $[0, 1]$.

Historically, population biologists were interested in those values of μ that give rise to stable populations after long term iteration. However, we shall see that as μ becomes close to 4, the long term behavior becomes highly unstable. The chaotic nature of this behavior was first pointed out by James Yorke in 1975. During a visit to Yorke at the University of Maryland, Robert May mentioned that he did not understand what happens to L_μ as μ approaches 4. Shortly after this, the seminal works of Li and Yorke ([**84**], 1975) and May ([**87**], 1976), appeared.



The logistic map with $\mu = 4$.

Remark 1.1.7 It is conjectured that closed form solutions for the difference equation arising from the logistic map are only possible when $\mu = -2$, $\mu = 2$ or $\mu = 4$ (see Exercises 1.1 # 3 for the cases where $\mu = 2$, $\mu = 4$ and 1.1 # 13 for the case where $\mu = -2$, and also [**128**] for a discussion of this conjecture).

Exercises 1.1

1. If $L_\mu(x) = \mu x(1-x)$ is the logistic map, calculate $L^2_\mu(x)$ and $L^3_\mu(x)$.
2. Use Example 1.1.3(3) for affine maps to find the solutions to the difference equations

   (i) $x_{n+1} - \dfrac{x_n}{3} = 2$, $x_0 = 2$,

   (ii) $x_{n+1} + 3x_n = 4$, $x_0 = -1$.
3. A *logistic difference equation* is one of the form $x_{n+1} = \mu x_n(1-x_n)$ for some fixed $\mu \in \mathbb{R}$. Find exact (closed form) solutions to the following logistic difference equations:

   (i) $x_{n+1} = 2x_n(1-x_n)$. (Hint: Use the substitution $x_n = (1-y_n)/2$ to transform the equation into a simpler equation that is easily solved.)

   (ii) $x_{n+1} = 4x_n(1-x_n)$. (Hint: Set $x_n = \sin^2\theta_n$ and simplify to get an equation that is easily solved.)
4. You borrow \$$P$ at r% per annum, and pay off \$$M$ at the end of each subsequent month. Write down a difference equation for the amount owing $A(n)$ at the end of each month (so $A(0) = P$). Solve the equation to find a closed form for $A(n)$. If $P = 100\,000$, $M = 1000$ and $r = 4$, after how long will the loan be paid off?
5. At 70.5 years of age, you have \$$A$ invested in a pre-tax retirement account. It is earning interest at r_1% per annum. The tax laws require you to take out r_2% per annum of what is remaining in the account ($r_2 > r_1$, where in practice $r_2 = 3.65$). How much is remaining after n years?

   Solve this problem with \$$A$ = \$500 000, $r_1 = 3\%$, $r_2 = 3.65\%$ and $n = 15$ years.
6. If $T_\mu(x) = \begin{cases} \mu x; & 0 \le x < 1/2, \\ \mu(1-x); & 1/2 \le x \le 1, \end{cases}$

   show that T_μ is a dynamical system of $[0,1]$, for $\mu \in (0,2]$.
7. Let $f(x) = x^2 + bx + c$. Give conditions on b and c for $f : [0,1] \to [0,1]$ to be a dynamical system. (Hint: Recall that that the maximum and minimum values of a continuous function defined on a closed interval

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$[a, b]$ occur at the end points, or where $f'(x) = 0$ or where $f'(x)$ does not exist.)

8. Determine whether the functions f defined below can be considered as dynamical systems $f : I \to I$:
   (a) $f(x) = x^3 - 3x$, (i) $I = [-1, 1]$, (ii) $I = [-2, 2]$.
   (b) $f(x) = 2x^3 - 6x$, (i) $I = [-1, 1]$, (ii) $I = \left[-\sqrt{\frac{7}{2}}, \sqrt{\frac{7}{2}}\right]$.
9. If $f_\mu(x) = \mu x^2 \dfrac{1-x}{1+x}$, show that for $0 < \mu < (5\sqrt{5} + 11)/2$, f_μ is a dynamical system of $[0, 1]$.
10. For the following functions, find $f^2(x)$, $f^3(x)$ and a general formula for $f^n(x)$:
    (i) $f(x) = x^2$, (ii) $f(x) = |x + 1|$, (iii) $f(x) = \begin{cases} 2x; & 0 \le x < 1/2, \\ 2x - 1; & 1/2 \le x < 1. \end{cases}$
11. Use mathematical induction to show that if $f(x) = \dfrac{2}{x+1}$, then
    $$f^n(x) = \frac{2^n(x+2) + (-1)^n(2x-2)}{2^n(x+2) - (-1)^n(x-1)}.$$
12. (a) The *tribonacci sequence* (T_n) is a generalization of the Fibonacci sequence, defined recursively by
    $$T_0 = 0,\ T_1 = 0,\ T_2 = 1, \quad T_{n+1} = T_n + T_{n-1} + T_{n-2}, \quad n \ge 2.$$
    Write down the first 10 terms of T_n.
    (b) Let (F_n) be the Fibonacci sequence. Set $v_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$, and $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Show that $v_{n+1} = F \cdot v_n$, $n \ge 0$.
    (c) Find a matrix T such that if (T_n) is the tribonacci sequence, and $w_n = \begin{pmatrix} T_{n+2} \\ T_{n+1} \\ T_n \end{pmatrix}$, then $w_{n+1} = T \cdot w_n$, $n \ge 0$.

13*. Show that a closed form solution to the logistic difference equation when $\mu = -2$ is given by

$$x_n = \frac{1}{2}\left[1 - f\left(r^n f^{-1}(1-2x_0)\right)\right], \quad \text{where} \quad r = -2 \quad \text{and}$$

$$f(\theta) = 2\cos\left(\frac{\pi - \sqrt{3}\theta}{3}\right).$$

(Hint: Set $x_n = \dfrac{1 - f(\theta_n)}{2}$ and use steps similar to those in Exercise 3(ii) above.)

1.2 Newton's Method and Fixed Points

Isaac Newton (1669) and Joseph Raphson (1690) gave special cases of what we now call Newton's method, with the modern version being given by Thomas Simpson in 1740. Newton's method is an algorithm for rapidly finding the approximate values of zeros of functions.

Given a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and under suitable conditions, Newton's method allows us to find good approximations to the zeros of $f(x)$, i.e., approximate solutions to the equation $f(x) = 0$. The idea is to start with a first approximation x_0, and look at the tangent line to $f(x)$ at the point $(x_0, f(x_0))$. Suppose this line intersects the x-axis at x_1; then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad \text{if} \quad f'(x_0) \neq 0.$$

If our initial guess x_0 is close enough to the zero, x_1 will be a better approximation to the zero. Repeat the process with the tangent line to $f(x)$ at $(x_1, f(x_1))$. At the $(n+1)$th stage we obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

an algorithm in the form of a difference equation, where x_0 is a first approximation to a zero of $f(x)$. The corresponding real function is

$$N_f(x) = x - \frac{f(x)}{f'(x)} \quad \text{(the \textit{Newton function})}.$$

For example, if $f(x) = x^2 - a$, then $f'(x) = 2x$ and

$$N_f(x) = x - \frac{x^2 - a}{2x} = \frac{1}{2}\left(x + \frac{a}{x}\right),$$