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978-1-107-11194-3 - Encyclopedia of Mathematics and its Applications: Variational Methods for
Nonlocal Fractional Problems

Giovanni Molica Bisci, Vicentiu D. Radulescu and Raffaella Servadei

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