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978-1-107-11194-3 - Encyclopedia of Mathematics and its Applications: Variational Methods for Nonlocal Fractional Problems

Giovanni Molica Bisci, Vicentiu D. Radulescu and Raffaella Servadei

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## Variational Methods For Nonlocal Fractional Problems

This book provides researchers and graduate students with a thorough introduction to the variational analysis of nonlinear problems described by nonlocal operators. The authors give a systematic treatment of the basic mathematical theory and constructive methods for these classes of nonlinear equations, plus their application to various processes arising in the applied sciences. The equations are examined from several viewpoints, with the calculus of variations as the unifying theme. Part I begins the book with some basic facts about fractional Sobolev spaces. Part II is dedicated to the analysis of fractional elliptic problems involving subcritical nonlinearities, via classical variational methods and other novel approaches. Finally, Part III contains a selection of recent results on critical fractional equations. A careful balance is struck between rigorous mathematics and physical applications, allowing readers to see how these diverse topics relate to other important areas, including topology, functional analysis, mathematical physics, and potential theory.

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# Variational Methods for Nonlocal Fractional Problems

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*“Io credo, per l’acume ch’io sofferesi  
del vivo raggio, ch’i’ sarei smarrito,  
se li occhi miei da lui fossero aversi”*

Dante, *La Divina Commedia*  
*Paradiso*, Canto XXXIII (vv. 76–78)

Dedicated to Antonella  
Bianca, Teodora  
and  
Antonietta, Enzo

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## Foreword

Although fractional derivatives for functions of one variable can be traced to the origin of calculus and were already introduced rigorously in the nineteenth century by Liouville and Riemann, the development of linear and nonlinear equations involving fractional derivatives of functions of one or several variables, and in particular fractional Laplacians, is a more recent phenomenon.

Rich mathematical concepts allow in general several approaches, and this is the case for the fractional Laplacian, which can be defined using Fourier analysis, functional calculus, singular integrals, or Lévy processes. Its inverse is closely related to the famous potentials introduced by Marcel Riesz in the late 1930s. In contrast to the Laplacian, which is a local operator, the fractional Laplacian is a paradigm of the vast family of nonlocal linear operators, and this has immediate consequences in the formulation of basic questions such as the Dirichlet problem.

If the Laplacian has been and still is a stimulating cornerstone of the theory of linear partial differential equations and linear operators, the twentieth century has seen an increasing and outstanding activity in the study of the Dirichlet or other boundary value problems for nonlinear perturbations of the Laplacian and for its quasi-linear or fully nonlinear extensions. All techniques of nonlinear functional analysis, such as iterative, topological, monotonicity, or variational methods, have been essentially tested on those problems and have received, at this occasion, important developments.

It is therefore a natural question to see which results “survive” when the Laplacian is replaced by the fractional Laplacian. It is also a fruitful question because the extension of classical results to new situations also sheds light on a better and deeper understanding of the classical results.

Giovanni Molica Bisci, Vicențiu Rădulescu, and Raffaella Servadei, the authors of this monograph, have all three, jointly or separately, contributed in a significant way to the use of modern critical point theory to nonlinear perturbations of a fractional Laplacian. But their monograph goes much beyond a presentation of

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their own results in book form. It starts with a substantial, clear, and systematic presentation of the fractional Sobolev spaces, where the considered problems will live, including the corresponding spectral theory. Then follows, for subcritical and critical perturbations, respectively, a panorama of how the methods of critical point theory apply successfully, at the expense of overcoming new technical difficulties, to many of the important problems considered in the Laplacian case.

For subcritical problems, mountain pass and linking-type nontrivial solutions are obtained, as well as Ricceri's solutions for parametric problems, followed by equations at resonance and the obtention of multiple solutions using pseudoindex theory. Kirchhoff-type and Schrödinger equations are also considered. For critical problems, emphasis is put on extending to the new setting the important results of Brezis–Nirenberg and related ones, as well as the case of concave-convex nonlinearities. One should notice that many results are obtained in situations where the fractional Laplacian is replaced by a more general nonlocal operator.

With its bibliography of some 200 items collecting the references to the original Laplacian perturbation results as well as to the corresponding fractional Laplacian extensions, and with its useful index, this carefully and clearly written monograph is and will remain a fundamental reference for any mathematician interested in the variational approach to nonlinear perturbations of nonlocal linear operators and, in particular, of the fractional Laplacian. Because those problems have been motivated by important applications, the book also will be useful to scientists interested in the mathematical techniques and ideas, allowing them to treat rigorously new models involving such nonlocal operators.

*Jean Mawhin, July 2015*

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## Preface

A very interesting area of nonlinear analysis lies in the study of elliptic equations involving fractional operators. Recently, great attention has been focused on these problems, both for pure mathematical research and in view of concrete real-world applications. Indeed, this type of operator arises in a quite natural way in different contexts, such as the description of several physical phenomena.

The current literature on these abstract tools and on their applications is therefore very interesting and, up to now, quite large. Motivated by this increasing interest on these subjects, this book deals with some classes of fractional problems widely investigated by many mathematicians and scientists.

This monograph is divided into three parts and is based on results obtained by ourselves or through direct cooperation with other mathematicians. More precisely, the first part deals with some basic facts about fractional Sobolev spaces, and the second part is dedicated to an analysis of fractional elliptic problems involving subcritical nonlinearities via classical variational methods and other novel approaches. Finally, in the third part of the book we give a selection of recent results on critical fractional equations, studied in the recent literature, also in relation to the celebrated Brezis–Nirenberg problem. Of course, there are many other interesting applications and theoretical aspects of fractional nonlocal problems, but it is not our ambition to treat all these topics here.

This book is addressed to researchers and advanced graduate students specializing in the fields of fractional elliptic equations, nonlinear analysis, and functional analysis. We also emphasize that the bibliography does not escape the usual rule, being incomplete. Indeed, we have listed only papers that are closer to the topics discussed in this book. But we are afraid that even for these arguments, the references are far from being exhaustive. We apologize for possible omissions.

We emphasize that this book would never have appeared without the encouragements of some dear friends and colleagues. It is a pleasure to thank some of them, especially Rossella Bartolo, Xavier Cabré, Philippe Ciarlet, Bernard Dacorogna,

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Alessio Fiscella, Jean Mawhin, Giuseppe Mingione, Giampiero Palatucci, Patrizia Pucci, Gaetana Restuccia, Biagio Ricceri, Dušan Repovš, Sandro Salsa, Simone Secchi, Francesco Tulone, Enrico Valdinoci, Gianmaria Verzini, and Binlin Zhang.

We also thank the colleagues and administrative staff of the Department Patrimonio, Architettura, Urbanistica of the *Mediterranea* University of Reggio Calabria for their help and the warm and creative atmosphere we always feel during our stay at that institution. A special mention goes to the Director Francesca Martorano for her extraordinary and constant friendship.

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