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# INTRODUCTION TO MODEL SPACES AND THEIR OPERATORS

The study of model spaces, the closed invariant subspaces of the backward shift operator, is a vast area of research with connections to complex analysis, operator theory, and functional analysis. This self-contained text is the ideal introduction for newcomers to the field. It sets out the basic ideas and quickly takes the reader through the history of the subject before ending up at the frontier of mathematical analysis. Open questions point to potential areas of future research, offering plenty of inspiration to graduate students wishing to advance further.

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# Introduction to Model Spaces and their Operators

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To our families:
Gizem; Reyhan, and Altay
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# Preface

This is an introductory text on model spaces that is aimed towards both graduate students and active researchers who wish to enter this evolving subject at a comfortable and digestible pace.

Model spaces have been studied, in one form or another, for the past 40 years, making connections to many areas of complex analysis (boundary behavior of analytic functions, analytic continuation, zero sets), operator theory (spectral properties, cyclic vectors, invariant subspaces, model operators for contractions, commutant lifting theorems, Hankel operators), engineering (the Darlington synthesis problem, control theory), and, more recently, to mathematical physics (completeness problems for Schrödinger and Sturm–Liouville operators). The purpose of this book is to present some of the basics of the subject in order to whet the appetite and to provide the newcomer with a solid foundation.

Many of the topics in this book were inspired by several series of lectures given by us at workshops in Montréal, Helsinki, Lens, Rennes, and Kashan where, during the course of these lectures, we became convinced of the need for students and researchers from adjacent fields to have a friendly introduction to model spaces and their operators.

This book is largely self-contained, although the reader is expected to be familiar with the basics of complex analysis, measure theory, and functional analysis. We briefly review these topics to establish our notation and, if necessary, to re-heat some possibly forgotten foundational topics. We develop and prove almost everything else, including a thorough treatment of the fundamentals of Hardy space theory, which is essential to the study of model spaces.

Since the list of topics we plan to cover is readily available in the table of contents, we would like to spend a few moments making the case for model spaces. Why should you keep on reading? Historically, model spaces began with the desire to characterize the cyclic vectors and invariant subspaces of the backward shift operator $S^*$ on the Hardy space $H^2$. Beurling's 1949 theorem completely characterized the non-trivial invariant subspaces of the unilateral shift $Sf = zf$ on $H^2$ as $uH^2$, where $u$ is an inner function on the open unit disk $\mathbb{D}$. By taking orthogonal complements, we see that the proper invariant subspaces of $S^*$ are the so-called model spaces $\mathcal{K}_u = (uH^2)^\perp$. The elements of $uH^2$ are easy to describe (multiples of the inner function $u$) while the elements of $\mathcal{K}_u$ are hidden behind annihilators and hence are more difficult to characterize. Indeed, which functions actually belong to $\mathcal{K}_u$?

In 1970, Douglas, Shapiro, and Shields linked membership to a model space with certain continuation (analytic and pseudocontinuation) properties of these functions. Around the same time, Ahern and Clark explored the close relationship between the boundary behavior of functions in $\mathcal{K}_u$ and the existence of angular derivatives, building upon earlier work of Carathéodory, Frostman, and Riesz.

Some of the most important theorems in operator theory involve modeling a class of abstract operators by concrete operators on familiar spaces. For example, there is the spectral theorem which models normal operators as multiplication operators on Lebesgue spaces. There are other representation theorems for subnormal operators and $n$-isometries in terms of multiplication operators on certain Hilbert spaces of analytic functions. Pushing this even further, there are the theorems of Sz.-Nagy and Foiaş which model certain types of contractions as the compression of the unilateral shift to a model space. This program was highly successful and gave reasons to broaden the study of model spaces from the scalar-valued case discussed above to the vector-valued case. The discussion was broadened even further with the discovery of the close cousins of the model spaces, the de Branges–Rovnyak spaces.

Sarason, in 1967, identified the commutant of the compressed shift. This result was greatly generalized by Sz.-Nagy and Foiaş to a wider class of operators and became known as the commutant lifting theorem – now regarded as one of the crowning gems of operator theory.

In 1972, Clark discovered a fascinating family of unitary operators whose associated spectral measures are ubiquitous in operator theory, complex analysis, and mathematical physics. These ideas were investigated further by Aleksandrov. Aleksandrov–Clark measures, as they have come to be known, have been used to study composition operators and are proving relevant in harmonic analysis. They also make connections to completeness problems for solutions to Schrödinger and Sturm–Liouville operators.

A seminal article of Sarason from 2008 initiated the study of truncated Toeplitz operators, close relatives of Toeplitz operators whose domains are model spaces. We discuss some of the foundational results of this evolving field, providing detailed proofs of the key results.

The topics mentioned above, as well as some others, are covered in this book along with all the necessary background material and historical references. Since this is an introduction to model spaces, we cannot cover everything. Although certain topics are missing, the topics that we do cover, we cover in great detail. We do not skimp on the explanations or examples. First and foremost, this book is meant to help the reader learn about model spaces and to become fluent with the fundamental ideas.

Finally, writing good books depends on valuable feedback from your colleagues. It this regard, we would like to thank John B. Conway, John E. McCarthy, Dan Timotin, and Dragan Vukotic for their comments on an earlier draft of this book. We also would like to thank Zachary Glassman for the wonderful drawings and Elizabeth Sarapata for the careful editing.

# Notation

| | |
|---|---|
| $\mathbb{C}$ | complex numbers |
| $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ | Riemann sphere |
| $\mathbb{C}_+$ | upper half plane |
| $\mathbb{C}_-$ | lower half plane |
| $\mathbb{N} = \{1, 2, \cdots\}$ | natural numbers |
| $\mathbb{D} = \{z \in \mathbb{C} : \|z\| < 1\}$ | open unit disk |
| $\mathbb{D}_e = \{z \in \mathbb{C} : \|z\| > 1\} \cup \{\infty\}$ | extended exterior disk |
| $\mathbb{T} = \{z \in \mathbb{C} : \|z\| = 1\}$ | unit circle |
| $A^-$ | the closure of $A$ |
| $M(\mathbb{T})$ | finite Borel measures on $\mathbb{T}$ (p. 1) |
| $M_+(\mathbb{T})$ | positive finite Borel measures on $\mathbb{T}$ (p. 1) |
| $m$ | normalized Lebesgue measure on $\mathbb{T}$ (p. 1) |
| $D\mu$ | symmetric derivative of a measure (p. 2) |
| $L^2 = L^2(\mathbb{T}, m)$ | standard Lebesuge space (p. 7) |
| $\widehat{f}(n)$ | Fourier coefficient of $f$ (p. 8) |
| $\ell^2(\mathbb{Z})$, $\ell^2(\mathbb{N})$ | square summable sequences (p. 8) |
| $P_z(\zeta)$ | Poisson kernel (p. 9) |
| $\mathscr{P}\mu$ | Poisson integral of a measure $\mu \in M(\mathbb{T})$ (p. 9) |
| $\widehat{\mu}(n)$ | Fourier coefficient of a measure (p. 10) |
| $\angle \lim_{z\to\zeta} f(z)$ | non-tangential limit of $f$ at $\zeta$ (p. 14) |
| $\mathcal{B}(\mathcal{H})$ | bounded operators on a Hilbert space $\mathcal{H}$ (p. 24) |
| $\sigma(A)$ | spectrum of an operator $A$ (p. 25) |
| $\sigma_p(A)$ | point spectrum of an operator $A$ (p. 25) |
| $\mathbf{x} \otimes \mathbf{y}$ | a rank one operator (p. 27) |
| $\sigma_e(A)$ | essential spectrum of an operator $A$ (p. 31) |
| $\tau_{\zeta,a}$ | disk automorphism (p. 32) |
| $H^\infty$ | bounded analytic functions on $\mathbb{D}$ (p. 33) |
| $H^2$ | Hardy space (p. 58) |

| | |
|---|---|
| $c_\lambda(z) = (1 - \overline{\lambda} z)^{-1}$ | reproducing kernel for $H^2$ (p. 59) |
| $P$ | Riesz projection onto $H^2$ (p. 66) |
| $S$ | unilateral shift on $H^2$ (p. 83) |
| $T_\varphi$ | Toeplitz operator on $H^2$ (p. 90) |
| $S^*$ | backward shift on $H^2$ (p. 99) |
| $\mathcal{K}_u = (uH^2)^\perp$ | model space (p. 104) |
| $Q_\lambda$ | difference quotient operator (p. 101) |
| $k_\lambda(z) = \frac{1-\overline{u(\lambda)}u(z)}{1-\overline{\lambda}z}$ | reproducing kernel for $\mathcal{K}_u$ (p. 111) |
| $P_u$ | projection onto $\mathcal{K}_u$ (p. 111) |
| $\mathscr{O}(\mathbb{D})$ | analytic functions on $\mathbb{D}$ (p. 126) |
| $C_\varphi$ | composition operator (p. 126) |
| $\sigma(u)$ | spectrum of an inner $u$ (p. 152) |
| $S_u$ | compressed shift (p. 189) |
| $\kappa_\lambda(z)$ | normalized reproducing kernel for $\mathcal{K}_u$ (p. 208) |
| $\sigma_\alpha$ | Clark measure (p. 232) |
| $U_\alpha$ | Clark unitary operator (p. 236) |
| $A^u_\varphi$ | truncated Toeplitz operator (p. 282) |
| $\mathscr{T}_u$ | space of truncated Toeplitz operators on $\mathcal{K}_u$ (p. 283) |