978-1-107-10874-5 — Introduction to Model Spaces and their Operators
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Preliminaries

1.1 Measure and integral

1.1.1 Borel sets and measures

Most of the "measuring" in this book will take place on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Since we assume that the reader has a background in graduate analysis, we quickly review the standard definitions without much fanfare.

We let $m := d\theta/2\pi$ denote *Lebesgue measure* on $\mathbb{T}$, normalized so that $m(\mathbb{T}) = 1$. A subset of $\mathbb{T}$ is called a *Borel set* if it is contained in the *Borel σ-algebra*, the smallest σ-algebra of subsets of $\mathbb{T}$ that contains all of the open arcs of $\mathbb{T}$. A *Borel measure* on $\mathbb{T}$ is a countably additive function that assigns a complex number to each Borel subset of $\mathbb{T}$. Unless otherwise stated, our measures will always be finite. A Borel measure is *positive* if it assigns a non-negative number to each Borel set. We let $M(\mathbb{T})$ denote the set of all complex Borel measures on $\mathbb{T}$ and we let $M_+(\mathbb{T})$ denote the set of all positive Borel measures on $\mathbb{T}$. A function $f : \mathbb{T} \to \widehat{\mathbb{C}}$ (where $\widehat{\mathbb{C}}$ denotes the Riemann sphere $\mathbb{C} \cup \{\infty\}$) satisfying the condition that $f^{-1}(U)$ is a Borel set for any open set $U \subset \widehat{\mathbb{C}}$ is called a *Borel function*.

We often need to distinguish between the "support" and a "carrier" of a measure. For $\mu \in M_+(\mathbb{T})$, consider the union $\mathcal{U}$ of all the open subsets $U \subset \mathbb{T}$ for which $\mu(U) = 0$. The complement $\mathbb{T} \setminus \mathcal{U}$ is called the *support* of μ. On the other hand, a Borel set $E \subset \mathbb{T}$ for which

$$\mu(E \cap A) = \mu(A) \tag{1.1}$$

for all Borel subsets $A \subset \mathbb{T}$ is called a *carrier of μ*. The support of μ is certainly a carrier, but a carrier need not be the support. Indeed, a carrier of a measure might not even be closed. For example, if $f \geqslant 0$ is continuous and $d\mu = f dm$, then a carrier of μ is $\mathbb{T} \setminus f^{-1}(\{0\})$ (which is open) while the support of μ is the closure of this set. The support of a measure is unique while a carrier is not.

The *Hahn–Jordan Decomposition Theorem* says that each $\mu \in M(\mathbb{T})$ can be written uniquely as

$$\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4), \qquad \mu_j \in M_+(\mathbb{T}), \tag{1.2}$$

in which μ_1, μ_2 and μ_3, μ_4, respectively, are carried on disjoint sets.

Since $\mathbb{T}$ is a compact Hausdorff space, each Borel measure μ on $\mathbb{T}$ is *regular* in the sense that each positive measure μ_j in the Hahn–Jordan Decomposition of μ satisfies

$$\inf\{\mu_j(U) : U \supset E, U \text{ open}\} = \sup\{\mu_j(F) : F \subset E, F \text{ closed}\} \tag{1.3}$$

for each Borel set $E \subset \mathbb{T}$ [158, p. 48]. Moreover, the quantity above is equal to $\mu_j(E)$.

Recall that $\mu \in M_+(\mathbb{T})$ is *absolutely continuous* with respect to m (written $\mu \ll m$) if $\mu(A) = 0$ whenever $m(A) = 0$. We say that μ is *singular* with respect to m (written $\mu \perp m$) if there are disjoint Borel sets A and B such that $\mathbb{T} = A \cup B$ and $\mu(A) = m(B) = 0$. Also recall that the Radon–Nikodym Theorem says that $\mu \ll m$ if and only if $d\mu = f dm$, where f is a Lebesgue integrable function on $\mathbb{T}$ (that is, $\int |f|\, dm < \infty$). By this we mean that $\mu(A) = \int_A f dm$ for each Borel set $A \subset \mathbb{T}$. The function f is unique up to a set of Lebesgue measure zero and is called the *Radon–Nikodym derivative* of μ (with respect to m). It is denoted by $d\mu/dm$. One can also obtain $d\mu/dm$ as a "derivative" as follows.

Definition 1.1 For $\mu \in M(\mathbb{T})$, the *symmetric derivative* $(D\mu)(w)$ of μ at $w \in \mathbb{T}$ is defined to be

$$(D\mu)(w) := \lim_{t \to 0^+} \frac{\mu((e^{-it}w, e^{it}w))}{m((e^{-it}w, e^{it}w))}, \tag{1.4}$$

whenever this limit exists. Here $(e^{-it}w, e^{it}w)$ denotes the arc of $\mathbb{T}$ subtended by the points $e^{-it}w$ and $e^{it}w$.

Theorem 1.2 *For each $\mu \in M(\mathbb{T})$, we have:*

(i) $(D\mu)(w)$ exists for m-almost every $w \in \mathbb{T}$ and

$$D\mu = \frac{d\mu}{dm}$$

m-almost everywhere;

(ii) $\mu \perp m$ if and only if $D\mu = 0$ m-almost everywhere;

(iii) If $\mu \in M_+(\mathbb{T})$ and $\mu \perp m$, then $D\mu = \infty$ μ-almost everywhere. Moreover, μ is carried by the set $\{\zeta : (D\mu)(\zeta) = \infty\}$.

The Lebesgue Decomposition Theorem says that every $\mu \in M(\mathbb{T})$ can be decomposed uniquely as

$$\mu = \mu_a + \mu_s, \tag{1.5}$$

where $\mu_a \ll m$ and $\mu_s \perp m$. The measure μ_a is called the *absolutely continuous part* of μ while μ_s is called the *singular part* of μ. Furthermore, the singular part μ_s can be decomposed as $\mu_s = \nu_d + \nu_c$, where

$$\nu_d = \sum_{n \geqslant 1} c_n \delta_{\zeta_n}$$

is a measure with distinct atoms at $\zeta_n \in \mathbb{T}$ (that is to say, for each Borel set $E \subset \mathbb{T}$, $\delta_{\zeta_n}(E) = 1$ if $\zeta_n \in E$ and zero otherwise) and weights $c_n = \nu_d(\{\zeta_n\})$, and where ν_c is a singular measure with no atoms (that is, $\nu_c(\{\zeta\}) = 0$ for all $\zeta \in \mathbb{T}$). The measure ν_d is called the *discrete part* of μ_s while ν_c is called the *singular continuous part* of μ_s. See below for a more classical approach to measures using functions of bounded variation.

We now review the *weak-∗ topology* on $M(\mathbb{T})$. Let $C(\mathbb{T})$ denote the algebra of complex-valued continuous functions on $\mathbb{T}$ endowed with the sup-norm

$$\|f\|_\infty := \sup_{\zeta \in \mathbb{T}} |f(\zeta)|.$$

Note that $C(\mathbb{T})$ is complete with respect to this norm and hence a Banach space. For each $\mu \in M(\mathbb{T})$, the linear functional

$$\ell_\mu : C(\mathbb{T}) \to \mathbb{C}, \qquad \ell_\mu(f) := \int_{\mathbb{T}} f \, d\mu$$

is bounded. The norm of ℓ_μ is defined by

$$\|\ell_\mu\| := \sup \{ |\ell_\mu(f)| : f \in C(\mathbb{T}) : \|f\|_\infty \leqslant 1 \}$$

and is equal to

$$\|\mu\| := |\mu|(\mathbb{T}), \tag{1.6}$$

where $|\mu|(\mathbb{T})$ is the supremum of $\sum_{n \geqslant 1} |\mu(E_n)|$ as $\{E_n\}_{n \geqslant 1}$ runs over all finite partitions of $\mathbb{T}$ into disjoint Borel subsets. The quantity $\|\mu\|$ is called the *total variation norm* of μ. In terms of the Hahn decomposition (1.2) of μ, it satisfies

$$\frac{1}{\sqrt{2}} \sum_{1 \leqslant j \leqslant 4} \mu_j(\mathbb{T}) \leqslant \|\mu\| \leqslant \sum_{1 \leqslant j \leqslant 4} \mu_j(\mathbb{T}).$$

Theorem 1.3 (Riesz Representation Theorem) *If ℓ is a bounded linear functional on $C(\mathbb{T})$, then $\ell = \ell_\mu$ for some unique $\mu \in M(\mathbb{T})$.*

This allows us to define the *weak-$*$ topology* on $M(\mathbb{T})$. A sequence $\{\mu_n\}_{n\geqslant 1} \subset M(\mathbb{T})$ converges *weak-$*$* to μ if

$$\lim_{n\to\infty} \int_{\mathbb{T}} f\, d\mu_n = \int_{\mathbb{T}} f\, d\mu \qquad \forall f \in C(\mathbb{T}). \tag{1.7}$$

The following tells us that closed balls in $M(\mathbb{T})$ are weak-$*$ compact.

Theorem 1.4 (Banach–Alaoglu) *If $\{\mu_n\}_{n\geqslant 1}$ is a sequence in $M(\mathbb{T})$ for which*

$$\sup_{n\geqslant 1} \|\mu_n\| < \infty,$$

then there is a measure $\mu \in M(\mathbb{T})$ and a subsequence μ_{n_k} that converges to μ in the weak-$*$ topology.

We will also need a version of the Hahn–Banach Theorem for $C(\mathbb{T})$ and $M(\mathbb{T})$.

Theorem 1.5 *Suppose that $\mathcal{M}$ is a linear manifold in $C(\mathbb{T})$ whose annihilator*

$$\left\{\mu \in M(\mathbb{T}) : \int_{\mathbb{T}} f d\mu = 0 \ \ \forall f \in \mathcal{M}\right\}$$

is zero. Then $\mathcal{M}$ is dense in $C(\mathbb{T})$. Furthermore, suppose that $\mathcal{N}$ is a linear manifold in $M(\mathbb{T})$ whose pre-annihilator

$$\left\{f \in C(\mathbb{T}) : \int_{\mathbb{T}} f d\mu = 0 \ \ \forall \mu \in \mathcal{N}\right\}$$

is zero. Then $\mathcal{N}$ is weak-$*$ dense in $M(\mathbb{T})$.

1.1.2 Classical approach to measures

The following classical approach to measure theory requires a discussion of functions of bounded variation and the Lebesgue–Stieltjes integral. We cover this material not only for students to reconnect with the classical roots of analysis, but to also help out with several proofs and examples later on.

Definition 1.6 A function $F : [0, 2\pi] \to \mathbb{C}$ is of *bounded variation* if

$$\|F\|_{BV} := \sup_{P} \sum_{0\leqslant j\leqslant n_P - 1} |F(x_{j+1}) - F(x_j)| < \infty, \tag{1.8}$$

where the supremum is taken over all partitions $P = \{x_0, x_1, \ldots, x_{n_P}\}$ of $[0, 2\pi]$, where $0 = x_0 < x_1 < x_2 \cdots < x_{n_P} = 2\pi$.

The expression (1.8) defines a semi-norm $\|F\|_{BV}$, the total variation (semi)-norm, on the set BV of all functions of bounded variation. Notice that $\|F\|_{BV}$ is not a true norm since $\|F\|_{BV} = 0$ if F is a constant function. We gather together some important facts about BV.

Proposition 1.7 *If $F \in BV$, then:*

(i) $F'(x)$ exists for m-almost every $x \in [0, 2\pi]$;
(ii) The one-sided limits

$$F(x^+) := \lim_{t\to x^+} F(t), \quad F(x^-) := \lim_{t\to x^-} F(t)$$

exist for every $x \in (0, 2\pi)$. Moreover, $F(0^+)$ and $F(2\pi^-)$ exist;
(iii) F has at most a countable number of discontinuities;
(iv) $F = (F_1 - F_2) + i(F_3 - F_4)$, where each F_j is increasing.

For $F \in BV$ and right continuous (that is, $F(x) = \lim_{t\to x^+} F(t)$ for all x), define μ_F on the set of half-open intervals

$$\{[a, b) : a, b \in [0, 2\pi], a \leqslant b\}$$

by

$$\mu_F([a, b)) := F(b) - F(a).$$

By the Carathéodory Extension Theorem, μ_F extends to a unique Borel measure on $[0, 2\pi]$. Moreover

$$\|\mu_F\| = \|F\|_{BV},$$

that is to say, the total variation norm of μ_F defined in (1.6) equals the bounded variation norm of F defined in (1.8). The integral

$$\int_{[0,2\pi]} f\, dF = \int_{[0,2\pi]} f\, d\mu_F, \tag{1.9}$$

defined for every $f \in C[0, 2\pi]$ (continuous functions on $[0, 2\pi]$), is called the *Lebesgue–Stieltjes integral* of f with respect to F.

In this classical setting, the Lebesgue Decomposition Theorem says that every $F \in BV$ can be written as

$$F = F_a + F_s,$$

where F_a is *absolutely continuous* (that is, f is the anti-derivative of a Lebesgue integrable function) on $[0, 2\pi]$ and F_s is *singular* (that is, $F_s' = 0$ almost everywhere with respect to Lebesgue measure). Note that $\mu_F = \mu_{F_a} + \mu_{F_s}$ is the

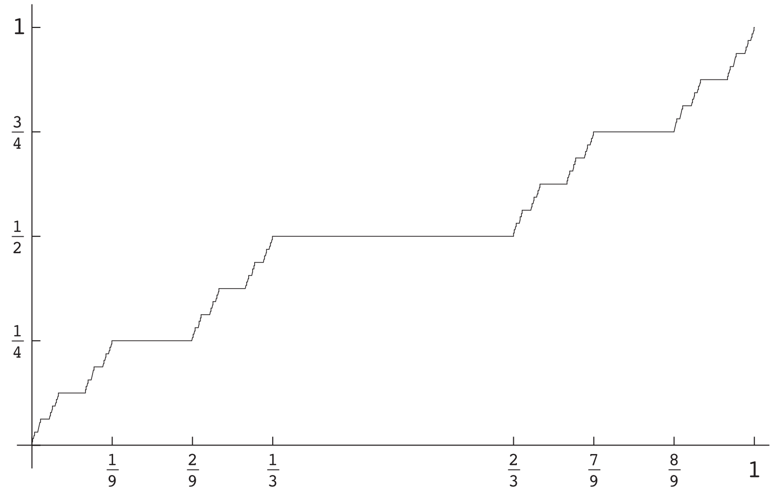

Figure 1.1 The Cantor devil's staircase function

decomposition of the measure μ_F into its absolutely continuous and singular parts from (1.5). Furthermore, F_s can be decomposed as

$$F_s = F_d + F_c,$$

where

$$F_d(x) = \sum_{y \leqslant x} (F(y^+) - F(y^-))$$

is a jump function and F_c is continuous with $F_c' = 0$ almost everywhere with respect to Lebesgue measure on $[0, 2\pi]$. This gives us the decomposition $\mu_{F_s} = \mu_{F_d} + \mu_{F_c}$ of μ_{F_s} into its discrete part μ_{F_d} and its continuous part μ_{F_c}.

For example, to produce a singular measure with no atoms one could take F to be the Cantor "devil's staircase" function (Figure 1.1). Note that F is continuous and $F' = 0$ almost everywhere with respect to Lebesgue measure on $[0, 2\pi]$. Thus $F = F_c$. The desired singular continuous measure is then μ_F.

The Riesz Representation Theorem tells us that every continuous linear functional on $C[0, 2\pi]$ takes the form

$$f \mapsto \int_{[0,2\pi]} f dF$$

for some unique (up to an additive constant) $F \in BV$. Moreover, the norm of this linear functional is $\|F\|_{BV}$.

The Banach–Alaoglu Theorem (Theorem 1.4) now takes the form of the *Helly Selection Theorem*: if $\{F_n\}_{n\geqslant 1} \subset BV$ and $\sup_{n\geqslant 1} \|F_n\|_{BV} < \infty$, then there is an $F \in BV$ and a subsequence F_{n_k} such that

$$\lim_{k\to\infty} \int_{[0,2\pi]} f\, dF_{n_k} = \int_{[0,2\pi]} f dF \qquad \forall f \in C[0,2\pi].$$

1.1.3 Lebesgue spaces

Let $L^2 := L^2(\mathbb{T}, m)$ denote the space of m-measurable (that is, Lebesgue measurable) functions $f : \mathbb{T} \to \widehat{\mathbb{C}}$ such that

$$\|f\| := \left(\int_{\mathbb{T}} |f|^2 dm\right)^{\frac{1}{2}} < \infty. \tag{1.10}$$

Retaining tradition, we equate two measurable functions which are equal almost everywhere. With this norm, L^2 is a Hilbert space endowed with the inner product

$$\langle f, g\rangle := \int_{\mathbb{T}} f\overline{g}\, dm.$$

From time to time we will need the spaces L^p $(0 < p < \infty)$ of measurable functions for which

$$\|f\|_p := \left(\int_{\mathbb{T}} |f|^p dm\right)^{\frac{1}{p}} < \infty.$$

We also require the space $L^\infty := L^\infty(\mathbb{T})$ of all essentially bounded measurable functions on $\mathbb{T}$, equipped with the essential supremum norm

$$\|f\|_\infty := \text{ess-sup}_{\zeta\in\mathbb{T}} |f(\zeta)|.$$

Here,

$$\text{ess-sup}_{\zeta\in\mathbb{T}} |f(\zeta)| := \sup\{a \geqslant 0 : m(\{\zeta \in \mathbb{T} : |f(\zeta)| > a\}) > 0\}$$

is the *essential supremum* of $|f|$.

For $\mu \in M_+(\mathbb{T})$, we will also need the corresponding $L^p(\mu)$ $(0 < p < \infty)$ spaces of Borel measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{L^p(\mu)} := \left(\int_{\mathbb{T}} |f|^p d\mu\right)^{\frac{1}{p}} < \infty.$$

The identity

$$\int_{\mathbb{T}} \zeta^n\, dm(\zeta) = \int_{[0,2\pi]} e^{in\theta} \frac{d\theta}{2\pi} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1.11}$$

shows that the family of functions $\{\zeta \mapsto \zeta^n : n \in \mathbb{Z}\}$ is an orthonormal set in L^2. The coefficients

$$\widehat{f}(n) := \langle f, \zeta^n \rangle = \int_{\mathbb{T}} f(\zeta)\overline{\zeta}^n \, dm(\zeta)$$

of an $f \in L^2$ with respect to this orthonormal set are called the (complex) *Fourier coefficients* of f.

Theorem 1.8 (Parseval's Theorem) *For each* $f \in L^2$,

$$\|f\|^2 = \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2.$$

Furthermore,

$$\lim_{N \to \infty} \Big\| f - \sum_{-N \leqslant n \leqslant N} \widehat{f}(n)\zeta^n \Big\| = 0.$$

The previous theorem tells us several things. First, $\{\zeta^n : n \in \mathbb{Z}\}$ is an orthonormal basis for L^2. Second, the *Fourier series*

$$\sum_{n \in \mathbb{Z}} \widehat{f}(n)\zeta^n$$

converges to f in the norm of L^2. In general, the Fourier series of an L^2 function need not converge pointwise. However, a deep theorem of Carleson says that it converges pointwise m-a.e. to f [17]. Although we will not use this fact, the reader should be aware that such delicate matters exist. Finally, Theorem 1.8 tells us that the L^2 norm of f coincides with the norm of the sequence $\{\widehat{f}(n) : n \in \mathbb{Z}\}$ of Fourier coefficients in the Hilbert space

$$\ell^2(\mathbb{Z}) := \Big\{ \mathbf{a} = \{a_n\}_{n \in \mathbb{Z}} : \|\mathbf{a}\|_{\ell^2(\mathbb{Z})} := \Big(\sum_{n \in \mathbb{Z}} |a_n|^2 \Big)^{\frac{1}{2}} < \infty \Big\}$$

of all square-summable sequences of complex numbers, endowed with the inner product

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{n \in \mathbb{Z}} a_n \overline{b_n}.$$

We therefore identify the Hilbert spaces L^2 and $\ell^2(\mathbb{Z})$ via the correspondence

$$f \leftrightarrow \{\widehat{f}(n) : n \in \mathbb{Z}\}.$$

1.2 Poisson integrals

The function

$$P_z(\zeta) := \frac{1 - |z|^2}{|\zeta - z|^2}, \qquad \zeta \in \mathbb{T}, \; z \in \mathbb{D},$$

is called the *Poisson kernel* of the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Note that

$$P_z(\zeta) > 0.$$

A computation verifies that

$$P_z(\zeta) = \operatorname{Re}\left(\frac{\zeta + z}{\zeta - z}\right) \tag{1.12}$$

and a computation with geometric series yields

$$P_{rw}(\zeta) = \sum_{n \in \mathbb{Z}} r^{|n|} w^n \overline{\zeta}^n, \quad w \in \mathbb{T}, \ r \in (0, 1). \tag{1.13}$$

With $w = e^{i\theta}$ and $\zeta = e^{it}$, one can also establish the formula

$$P_{re^{i\theta}}(e^{it}) = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}. \tag{1.14}$$

For fixed $\zeta \in \mathbb{T}$, $P_z(\zeta)$ is the real part of the analytic function

$$z \mapsto \frac{\zeta + z}{\zeta - z},$$

which makes $z \mapsto P_z(\zeta)$ a harmonic function on $\mathbb{D}$. Integrating the series in (1.13) term by term and using the orthogonality relations (1.11), we see that

$$\int_{\mathbb{T}} P_z(\zeta)\, dm(\zeta) = 1, \qquad z \in \mathbb{D}. \tag{1.15}$$

An important property of the Poisson kernel is that for fixed $\delta > 0$,

$$\lim_{r \to 1^-} \Big(\sup_{\delta \leqslant |t| \leqslant \pi} P_r(e^{it}) \Big) = 0. \tag{1.16}$$

This is illustrated in Figure 1.2. One can also see this from the estimate

$$P_r(e^{it}) \leqslant \frac{1 - r^2}{1 - 2\cos\delta + r^2}, \quad \delta \leqslant |t| \leqslant \pi.$$

For $\mu \in M(\mathbb{T})$, define the *Poisson integral of* μ by

$$\mathscr{P}(\mu)(z) := \int_{\mathbb{T}} P_z(\zeta) d\mu(\zeta), \quad z \in \mathbb{D}.$$

By differentiating under the integral sign, we see that $\mathscr{P}(\mu)$ is harmonic on $\mathbb{D}$. Furthermore,

$$\mathscr{P}(\mu)(rw) = \sum_{n \in \mathbb{Z}} \widehat{\mu}(n) r^{|n|} w^n, \quad w \in \mathbb{T}, \ r \in (0, 1), \tag{1.17}$$

where

$$\widehat{\mu}(n) := \int_{\mathbb{T}} \overline{\zeta}^n d\mu(\zeta), \quad n \in \mathbb{Z},$$

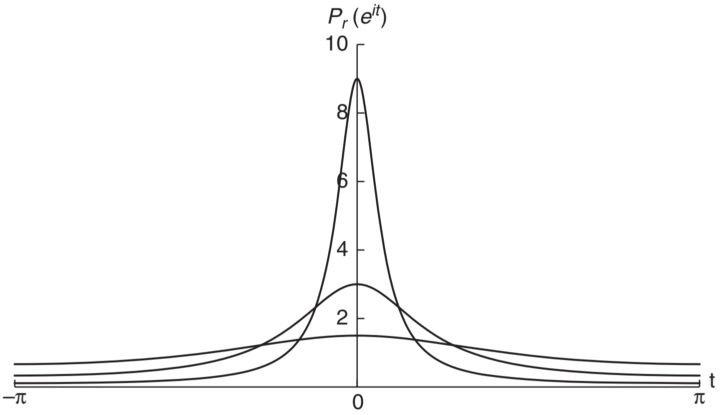

Figure 1.2 The graphs of $P_r(e^{it})$ for $r = 0.2, 0.5, 0.8$. Notice that $P_r(e^{it}) > 0$. Furthermore, notice how P_r peaks higher and higher near $t = 0$, for increasing values of r, while decaying rapidly away from the origin.

are the *Fourier coefficients* of the measure μ. We often write $\mathscr{P}(f)$ in place of the more cumbersome $\mathscr{P}(f dm)$ for $f \in L^1$.

For $f \in C(\mathbb{T})$, we have the Poisson Integral Formula for the solution of the Dirichlet problem on $\mathbb{D}^-$. The classical Dirichlet problem for a planar domain Ω is: given a continuous function f on the boundary $\partial\Omega$ of Ω, find a function u which is continuous on Ω^- that is harmonic on Ω and agrees with f on $\partial\Omega$.

Theorem 1.9 (Poisson Integral Formula) *If $f \in C(\mathbb{T})$, then $\mathscr{P}(f)$ is harmonic on $\mathbb{D}$ and extends continuously to $\mathbb{D}^-$. Furthermore,*

$$\lim_{z \to w} \mathscr{P}(f)(z) = f(w)$$

for every $w \in \mathbb{T}$.

Proof Let $u := \mathscr{P}(f)$ and recall from our earlier discussion that u is harmonic on $\mathbb{D}$. To complete the proof, we will show that for every fixed $w \in \mathbb{T}$,

$$\lim_{z \to w} u(z) = f(w). \tag{1.18}$$

Let $\varepsilon > 0$ be given. Use the continuity of f at w to produce a $\delta > 0$ so that whenever $\zeta \in \mathbb{T}$ and $|w - \zeta| < \delta$ we have $|f(w) - f(\zeta)| < \varepsilon$. From here we get

$$\begin{aligned} |u(z) - f(w)| &= \left| \int_{\mathbb{T}} f(\zeta) P_z(\zeta) dm(\zeta) - f(w) \int_{\mathbb{T}} P_z(\zeta) dm(\zeta) \right| \qquad \text{(by (1.15))} \\ &\leqslant \int_{\mathbb{T}} |f(\zeta) - f(w)| P_z(\zeta) dm(\zeta) \end{aligned}$$