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## FOURIER ANALYSIS AND HAUSDORFF DIMENSION

During the past two decades there has been active interplay between geometric measure theory and Fourier analysis. This book describes part of that development, concentrating on the relationship between the Fourier transform and Hausdorff dimension.

The main topics concern applications of the Fourier transform to geometric problems involving Hausdorff dimension, such as Marstrand type projection theorems and Falconer's distance set problem, and the role of Hausdorff dimension in modern Fourier analysis, especially in Keane methods and Fourier restriction phenomena. The discussion includes both classical results and recent developments in the area. The author emphasizes partial results of important open problems, for example, Falconer's distance set conjecture, the Keane conjecture and the Fourier restriction conjecture. Essentially self-contained, this book is suitable for graduate students and researchers in mathematics.

**Pertti Mattila** is Professor of mathematics at the University of Helsinki and an expert in geometric measure theory. He has authored the book *Geometry of Sets and Measures in Euclidean Spaces* as well as more than 80 other scientific publications.

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# Fourier Analysis and Hausdorff Dimension

PERTTI MATTILA  
*University of Helsinki*



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To John Marstrand

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## Preface

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This is a book on geometric measure theory and Fourier analysis. The main purpose is to present several topics where these areas meet including some of the very active recent interplay between them. We shall essentially restrict ourselves to questions involving the Fourier transform and Hausdorff dimension leaving many other aspects aside.

The book is intended for graduate students and researchers in mathematics. The prerequisites for reading it are basic real analysis and measure theory. Familiarity with Hausdorff measures and dimension and with Fourier analysis is certainly useful, but all that is needed will be presented in Chapters 2 and 3. Although most of the material has not appeared in book form, there is overlap with several earlier books. In particular, Mattila [1995] covers part of Chapters 4–7, Wolff [2003] of Chapters 14, 19, 20 and 22, and Stein [1993] of 14 and 19–21. Several other overlaps are mentioned in the text. The surveys Iosevich [2001], Łaba [2008], [2014], Mattila [2004], Mitsis [2003a] and Tao [2001], [2004] are closely related to the themes of the book.

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