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Introduction

The main object of this book is the interplay between geometric measure theory and Fourier analysis on $\mathbb{R}^n$. The emphasis will be more on the first in the sense that on several occasions we look for the best known results in geometric measure theory while our goals in Fourier analysis will usually be much more modest. We shall concentrate on those parts of Fourier analysis where Hausdorff dimension plays a role. Much more between geometric measure theory and Fourier analysis has been and is going on. Relations between singular integrals and rectifiability have been intensively studied for more than two decades; see the books David and Semmes [1993], Mattila [1995] and Tolsa [2014], the survey Volberg and Eiderman [2013], and Nazarov, Tolsa and Volberg [2014] for recent break-through results. Relations between harmonic measure, partial differential equations (involving a considerable amount of Fourier analysis) and rectifiability have recently been very actively investigated by many researchers; see, for example, Kenig and Toro [2003], Hofmann, Mitrea and Taylor [2010], Hofmann, Martell and Uriarte-Tuero [2014], and the references given therein.

In this book there are two main themes. Firstly, the Fourier transform is a powerful tool on geometric problems concerning Hausdorff dimension, and we shall give many applications. Secondly, some basic problems of modern Fourier analysis, in particular those concerning restriction, are related to geometric measure theoretic Kakeya (or Besicovitch) type problems. We shall discuss these in the last part of the book. We shall also consider various particular constructions of measures and the behaviour of their Fourier transforms.

The contents of this book can be divided into four parts.

PART I Preliminaries and some simpler applications of the Fourier transform.

PART II Specific constructions.

PART III Deeper applications of the Fourier transform.

PART IV Fourier restriction and Kakeya type problems.

Parts I and III are closely linked together. They are separated by Part II only because much of the material in Part III is rather demanding and Part II might be more easily digestible. In any case, the reader may jump over Part II without any problems. On the other hand, the sections of Part II are essentially independent of each other and only rely on Chapters 2 and 3. Part IV is nearly independent of the others. In addition to the basics of the Fourier transform, given in Chapter 3, the reader is advised to consult Chapter 11 on Besicovitch sets and Chapter 14 on oscillatory integrals before reading Part IV.

The applicability of the Fourier transform on Hausdorff dimension stems from the following three facts. First, the Hausdorff dimension of a Borel set $A \subset \mathbb{R}^n$, $\dim A$, can be determined by looking at the behaviour of Borel measures μ with compact support $\operatorname{spt}\mu \subset A$. We denote by $\mathcal{M}(A)$ the family of such measures μ with $0 < \mu(A) < \infty$. More precisely, by Frostman's lemma $\dim A$ is the supremum of the numbers s such that there exists $\mu \in \mathcal{M}(A)$ for which

$$\mu(B(x,r)) \leq r^s \quad \text{for } x \in \mathbb{R}^n, \quad r > 0. \tag{1.1}$$

This is easily transformed into an integral condition. Let

$$I_s(\mu) = \iint |x-y|^{-s}\, d\mu x\, d\mu y$$

be the s-energy of μ. Then $\dim A$ is the supremum of the numbers s such that there exists $\mu \in \mathcal{M}(A)$ for which

$$I_s(\mu) < \infty. \tag{1.2}$$

For a given μ the conditions (1.1) and (1.2) may not be equivalent, but they are closely related: (1.2) implies that the restriction of μ to a suitable set with positive μ measure satisfies (1.1), and (1.1) implies that μ satisfies (1.2) for any $s' < s$. Defining the Riesz kernel k_s, $k_s(x) = |x|^{-s}$, the s-energy of μ can be written as

$$I_s(\mu) = \int k_s * \mu\, d\mu.$$

For $0 < s < n$ the Fourier transform of k_s (in the sense of distributions) is $\widehat{k_s} = \gamma(n,s)k_{n-s}$ where $\gamma(n,s)$ is a positive constant. Thus we have by Parseval's theorem

$$I_s(\mu) = \int \widehat{k_s}|\widehat{\mu}|^2 = \gamma(n,s)\int |x|^{s-n}|\widehat{\mu}(x)|^2\, dx.$$

Consequently, $\dim A$ is the supremum of the numbers s such that there exists $\mu \in \mathcal{M}(A)$ for which

$$\int |x|^{s-n} |\widehat{\mu}(x)|^2 \, dx < \infty. \tag{1.3}$$

Thus, in a sense, a large part of this book is a study of measures satisfying one, or all, of the conditions (1.1), (1.2) or (1.3). As we shall see, in many applications using (1.1) or (1.2) is enough but often (1.3) is useful and sometimes indispensable. In the most demanding applications one has to go back and forth with these conditions.

The first application of Fourier transforms to Hausdorff dimension was Kaufman's [1968] proof for one part of Marstrand's projection theorem. This result, proved by Marstrand [1954], states the following.

Suppose $A \subset \mathbb{R}^2$ is a Borel set and denote by $P_e, e \in S^1$, the orthogonal projection onto the line $\{te : t \in \mathbb{R}\}$: $P_e(x) = e \cdot x$.
(1) If $\dim A \leq 1$, then $\dim P_e(A) = \dim A$ for almost all $e \in S^1$.
(2) If $\dim A > 1$, then $\mathcal{L}^1(P_e(A)) > 0$ for almost all $e \in S^1$.

Here $\mathcal{L}^1$ is the one-dimensional Lebesgue measure.

Marstrand's original proof was based on the definition and basic properties of Hausdorff measures. Kaufman used the characterization (1.2) for the first part and (1.3) for the second part. We give here Kaufman's proof to illustrate the spirit of the techniques used especially in Part I; many of the later arguments are variations of the following.

To prove (1) let $0 < s < \dim A$ and choose by (1.2) a measure $\mu \in \mathcal{M}(A)$ such that $I_s(\mu) < \infty$. Let $\mu_e \in \mathcal{M}(P_e(A))$ be the push-forward of μ under P_e: $\mu_e(B) = \mu(P_e^{-1}(B))$. Then

$$\begin{aligned}\int_{S^1} I_s(\mu_e)\, de &= \int_{S^1} \iint |e \cdot (x-y)|^{-s} \, d\mu x \, d\mu y \, de \\ &= \iiint_{S^1} |e \cdot (\tfrac{x-y}{|x-y|})|^{-s} \, de |x-y|^{-s} \, d\mu x \, d\mu y = c(s) I_s(\mu) < \infty,\end{aligned}$$

where for $v \in S^1$, $c(s) = \int_{S^1} |e \cdot v|^{-s} \, de < \infty$ as $s < 1$. Referring again to (1.2) we see that $\dim P_e(A) \geq s$ for almost all $e \in S^1$. By the arbitrariness of $s, 0 < s < \dim A$, we obtain $\dim P_e(A) \geq \dim A$ for almost all $e \in S^1$. The opposite inequality follows from the fact that the projections are Lipschitz mappings.

To prove (2) choose by (1.3) a measure $\mu \in \mathcal{M}(A)$ such that $\int |x|^{-1} |\widehat{\mu}(x)|^2 \, dx < \infty$. Directly from the definition of the Fourier transform we see that $\widehat{\mu_e}(t) = \widehat{\mu}(te)$ for $t \in \mathbb{R}, e \in S^1$. Integrating in polar coordinates we obtain

$$\int_{S^1}\int_{-\infty}^{\infty}|\widehat{\mu_e}(t)|^2\,dt\,de = 2\int_{S^1}\int_0^{\infty}|\widehat{\mu}(te)|^2\,dt\,de = 2\int|x|^{-1}|\widehat{\mu}(x)|^2\,dx < \infty.$$

Thus for almost all $e \in S^1$, $\widehat{\mu_e} \in L^2(\mathbb{R})$ which means that μ_e is absolutely continuous with L^2 density and hence $\mathcal{L}^1(P_e(A)) > 0$.

The interplay between geometric measure theory and Fourier restriction, that we shall discuss in Part IV, has its origins in the following observations:

Let g be a function on the unit sphere S^{n-1}, for example the restriction of the Fourier transform of a smooth function f defined on $\mathbb{R}^n$. Let us fatten the sphere to a narrow annulus $S(\delta) = \{x : 1-\delta < |x| < 1+\delta\}$. We can write this annulus as a disjoint union of $\approx \delta^{(1-n)/2}$ spherical caps R_j, each of which is almost (for a small δ) a rectangular box with $n-1$ side-lengths about $\sqrt{\delta}$ and one about δ. Suppose we could write $g = \sum_j g_j$ where each g_j is a smooth function with compact support in R_j (which of course we usually cannot do). Then $f = \sum_j f_j$ where f_j is the inverse transform of g_j, which is almost the same as the Fourier transform of g_j. A simple calculation reveals that f_j is like a smoothened version of the characteristic function of a dual rectangular box $\widetilde{R_j}$ of R_j; it decays very fast outside $\widetilde{R_j}$. This dual rectangular box is a rectangular box with $n-1$ side-lengths about $1/\sqrt{\delta}$ and one about $1/\delta$, so it is like a long narrow tube. Thus studying f based on the information we have about the restriction of its Fourier transform on S^{n-1}, we are lead to study huge collections of narrow tubes and the behaviour of sums of functions essentially supported on them. These are typical Kakeya problems.

A concrete result along these lines is:

If the restriction conjecture is true, then all Besicovitch sets in $\mathbb{R}^n$ have Hausdorff dimension n.

The restriction conjecture, or one form of it, says that the Fourier transform of any function in $L^p(\mathbb{R}^n)$ can be meaningfully restricted to S^{n-1} when $1 \leq p < \frac{2n}{n+1}$. In the dual form this amounts to saying that the Fourier transform defines a bounded operator $L^{\infty}(S^{n-1}) \to L^q(\mathbb{R}^n)$ for $q > \frac{2n}{n-1}$ in the sense that the inequality

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^{\infty}(S^{n-1})}$$

holds. For $n = 2$ this is known to be true and for $n > 2$ it is open. The restriction conjecture is related to many other questions of modern Fourier analysis and partial differential equations. We shall discuss some of these in this book.

Besicovitch sets are sets of Lebesgue measure zero containing a unit line segment in every direction. They exist in $\mathbb{R}^n$ for all $n \geq 2$. It is known, and we shall prove it, that all Besicovitch sets in the plane have Hausdorff dimension 2, but in higher dimensions it is an open problem whether they have full dimension n. Fattening Besicovitch sets slightly we end up with collections of narrow tubes as discussed above.

Now I give a brief overview of each chapter. Chapter 2 gives preliminaries on Borel measures in $\mathbb{R}^n$ and Chapter 3 on the Fourier transform, including the proofs for the characterization of Hausdorff dimension in terms of (1.1), (1.2) and (1.3). In Chapter 4 we repeat the above proof for Marstrand's theorem with more details and study Falconer's distance set problem: what can we say about the size of the distance set

$$D(A) = \{|x - y| : x, y \in A\}$$

if we know the Hausdorff dimension of a Borel set $A \subset \mathbb{R}^n$? For instance, we show that if $\dim A > (n+1)/2$ then $D(A)$ contains an open interval. In Chapter 5 we sharpen Marstrand's projection theorem by showing that the Hausdorff dimension of the exceptional directions in (1) is at most $\dim A$ and in (2) at most $2 - \dim A$. We also give the higher dimensional versions and introduce the concept of Sobolev dimension of a measure, the use of which unifies and extends the results. In Chapter 6 we slice, or disintegrate, Borel measures in $\mathbb{R}^n$ by m-planes and apply this process to prove that typically if an m-plane V intersects a Borel set $A \subset \mathbb{R}^n$ with $\dim A > n - m$, it intersects it in dimension $\dim A + m - n$. We also prove here an exceptional set estimate and give an application to the Fourier transforms of measures on graphs. Chapter 7 studies the more general question of generic intersections of two arbitrary Borel sets. We prove that if $A, B \subset \mathbb{R}^n$ are Borel sets and $\dim B > (n+1)/2$, then for almost all rotations $g \in O(n)$ the set of translations by $z \in \mathbb{R}^n$ such that $\dim A \cap (g(B) + z) \geq \dim A + \dim B - n - \varepsilon$ has positive Lebesgue measure for every $\varepsilon > 0$.

We start Part II by studying in Chapter 8 classical symmetric Cantor sets with dissection ratio d and the natural measures on them. We compute the Fourier transform and show that it goes to zero at infinity if and only if $1/d$ is not a Pisot number. Bernoulli convolutions are studied in Chapter 9. They are probability distributions of random sums $\sum_j \pm\lambda^j$, $0 < \lambda < 1$. We prove part of Solomyak's theorem which says that they are absolutely continuous for almost all $\lambda \in (1/2, 1)$. In Chapter 10 we investigate projections of the one-dimensional Cantor set in the plane which is the product of two standard symmetric linear half-dimensional Cantor sets. We show in two ways that it projects into a set of Lebesgue measure zero on almost all lines and we also derive more detailed information about its projections. Using the aforementioned result we construct Besicovitch sets in Chapter 11. We shall also prove there that they have Hausdorff dimension at least 2. We shall consider Nikodym sets, too. They are sets of measure zero containing a line segment on some line through every point of the space. In Chapter 12 we find sharp information about the almost sure decay of Fourier transforms of some measures on trajectories of Brownian motion. The decay is as fast as the Hausdorff dimension allows, so the trajectories give examples of Salem sets. In Chapter 13 we study absolute continuity properties, both with respect to Lebesgue measure and Hausdorff dimension, of classical Riesz products. In Chapter 14 we derive basic decay properties for oscillatory integrals $\int e^{i\lambda\varphi(x)}\psi(x)\,dx$ and apply them to the Fourier transform of some surface measures.

Beginning Part III in Chapter 15 we return to the applications of Fourier transforms to geometric problems on Hausdorff dimension; we apply decay estimates of the spherical averages $\int_{S^{n-1}} |\widehat{\mu}(rv)|^2\,dv$ to distance sets. We continue this in Chapter 16 and prove deep estimates of Wolff and Erdoğan using Tao's bilinear restriction theorem (which is proved later) and Kakeya type methods. This will give us the best known dimension results for the distance set problem. In Chapter 17 we define fractional Sobolev spaces in terms of Fourier transforms. We study convergence questions for Sobolev functions and for solutions of the Schrödinger equation and estimate the Hausdorff dimension of the related exceptional sets. The Fourier analytic techniques of Peres and Schlag are introduced in Chapter 18 and they are applied to get considerable extensions of projection type theorems, both in terms of mappings and in terms of exceptional set estimates.

In Part IV we first introduce in Chapter 19 the restriction problems and prove the basic Stein–Tomas theorem. It says that

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^2(S^{n-1})} \quad \text{for } q \geq 2(n+1)/(n-1).$$

In fact, we do not prove the end-point estimate for $q = 2(n+1)/(n-1)$, but we shall give a sketch for it in Chapter 20 using a stationary phase method. We shall also prove the restriction conjecture

$$\|\widehat{f}\|_{L^q(\mathbb{R}^2)} \leq C(q)\|f\|_{L^\infty(S^1)} \quad \text{for } q > 4$$

in the plane using this method.

In Chapter 21 we first prove Fefferman's multiplier theorem saying that for a ball B in $\mathbb{R}^n$, $n \geq 2$, the multiplier operator T_B, $\widehat{T_B f} = \chi_B \widehat{f}$, is not bounded in L^p if $p \neq 2$. This uses Kakeya methods and really is the origin for the applications of such methods in Fourier analysis. We shall also briefly discuss Bochner–Riesz multipliers. In Chapter 22 we introduce the Kakeya maximal function

$$\mathcal{K}_\delta f : S^{n-1} \to [0, \infty],$$

$$\mathcal{K}_\delta f(e) = \sup_{a \in \mathbb{R}^n} \frac{1}{\mathcal{L}^n(T_e^\delta(a))} \int_{T_e^\delta(a)} |f| \, d\mathcal{L}^n$$

and study its mapping properties. Here $T_e^\delta(a)$ is a tube of width δ and length 1, with direction e and centre a. The Kakeya maximal conjecture is

$$\|\mathcal{K}_\delta f\|_{L^n(S^{n-1})} \leq C_\varepsilon \delta^{-\varepsilon} \|f\|_{L^n(\mathbb{R}^n)} \text{ for all } \varepsilon > 0,\ f \in L^n(\mathbb{R}^n).$$

We shall prove that it follows from the restriction conjecture and implies the Kakeya conjecture that all Besicovitch sets in $\mathbb{R}^n$ have Hausdorff dimension n. We shall also show that the analogue of the Kakeya conjecture is true in the discrete setting of finite fields.

In Chapter 23 we prove various estimates for the Hausdorff dimension of Besicovitch sets. In particular, we prove Wolff's lower bound $(n+2)/2$ with geometric methods and the Bourgain–Katz–Tao lower bound $6n/11 + 5/11$ with arithmetic methods. In Chapter 24 we study (n, k) Besicovitch sets; sets of measure zero containing a positive measure piece of a k-plane in every direction. Following Marstrand and Falconer we first give rather simple proofs that they do not exist if $k > n/2$. Then we shall present Bourgain's proof which relies on Kakeya maximal function inequalities and extends this to $k > (n+1)/3$, and even further with more complicated arguments which we shall only mention.

The last chapter, Chapter 25, gives a proof for Tao's sharp bilinear restriction theorem:

$$\|\widehat{f_1}\widehat{f_2}\|_{L^q(\mathbb{R}^n)} \lesssim \|f_1\|_{L^2(S^{n-1})}\|f_2\|_{L^2(S^{n-1})} \quad \text{for } q > (n+2)/n,$$

when $f_j \in L^2(S^{n-1})$ with dist(spt f_1, spt f_2) ≈ 1. In fact, we shall prove a weighted version of this due to Erdoğan which is needed for the aforementioned distance set theorem. We shall also deduce a partial result for the restriction conjecture from this bilinear estimate.

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More information

PART I

Preliminaries and some simpler applications of the Fourier transform