QUANTUM GRAVITY AND THE FUNCTIONAL RENORMAIZATION GROUP

The Road towards Asymptotic Safety

During the past two decades the gravitational Asymptotic Safety scenario has undergone a major transition from an exotic possibility to a serious contender as a realistic theory of quantum gravity. It aims at a mathematically consistent quantum description of the gravitational interaction and the geometry of spacetime within the realm of quantum field theory, keeping its predictive power at the highest energies. This volume provides a self-contained pedagogical introduction to Asymptotic Safety and introduces the functional renormalization group techniques used in its investigation, along with the requisite computational techniques. The foundational chapters are followed by an accessible summary of the results obtained thus far. It is the first detailed exposition of Asymptotic Safety, providing a unique introduction to quantum gravity. The text assumes no previous familiarity with the renormalization group and thus serves as an important resource for both practicing researchers and graduate students entering this maturing field.

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Preface

This book grew out of a series of lectures the authors have given at various universities and summer schools. Its intention is to provide an easily accessible, pedagogical account of the basic conceptual ideas and methods underlying the asymptotic safety approach to quantum gravity. Knowledge of General Relativity and quantum field theory at a master-course level should suffice to follow the exposition. The necessary technical background is developed from the beginning and no previous familiarity with functional renormalization group methods is assumed. As much as possible we try to supplement formal derivations by intuitive arguments. Our hope is that the book provides a valuable resource for graduate students and researchers, enabling them to follow the cutting-edge research in this field and placing it into the broad context.

It is impossible to thank all people here who directly or indirectly contributed to our work on quantum gravity and to this book ultimately. However, M. R. is particularly grateful to Christof Wetterich for the crucial and highly inspiring collaborations during the early days of the functional renormalization group and more than three decades of creative exchange and to Roberto Percacci for sharing the dream about asymptotic safety right from the start and for his decisive work toward making it a reality. It is also a pleasure to thank Alfio Bonanno for an enjoyable collaboration from very early on in an effort to understand the phenomenological consequences of asymptotic safety. Special heartfelt thanks go to Ennio Gozzi and Walter Dittrich for their support and guidance across all of physics, and well beyond.

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List of Abbreviations

ADM       Arnowitt–Deser–Misner
BRST      Becchi–Rouet–Stora–Tyutin
CDT       Causal Dynamical Triangulation
CREH      conformally reduced Einstein–Hilbert
EAA       Effective Average Action
EH        Einstein–Hilbert
FRG       Functional Renormalization Group
FRGE      Functional Renormalization Group Equation
GFP       Gaussian fixed point
GR        General Relativity
IR        infrared
LHS       left-hand side
LPA       local potential approximation
LQG       Loop Quantum Gravity
LSZ       Lehmann–Symanzik–Zimmermann
NGFP      non-Gaussian fixed point
ODE       ordinary differential equation
QCD       Quantum Chromodynamics
QED       Quantum Electrodynamics
QEG       Quantum Einstein Gravity
RG        renormalization group
RHS       right-hand side
UV        ultraviolet