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**SPECIAL FUNCTIONS AND ORTHOGONAL POLYNOMIALS**

The subject of special functions is often presented as a collection of disparate results, rarely organized in a coherent way. This book emphasizes general principles that unify and demarcate the subjects of study. The authors' main goals are to provide clear motivation, efficient proofs, and original references for all of the principal results.

The book covers standard material, but also much more. It shows how much of the subject can be traced back to two equations – the hypergeometric equation and the confluent hypergeometric equation – and it details the ways in which these equations are canonical and special. There is extended coverage of orthogonal polynomials, including connections to approximation theory, continued fractions, and the moment problem, as well as an introduction to new asymptotic methods. The book includes chapters on Meijer *G*-functions and elliptic functions. The final chapter introduces Painlevé transcendents, which have been termed the "special functions of the twenty-first century."

**Richard Beals** was Professor of Mathematics at the University of Chicago and at Yale University. He is the author or co-author of books on mathematical analysis, linear operators, and inverse scattering theory, and has authored more than a hundred research papers in areas including partial differential equations, mathematical economics, and mathematical psychology.

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# Special Functions and Orthogonal Polynomials

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# Contents

# Preface

This book originated as *Special Functions: A Graduate Text*. The current version is considerably enlarged: the number of chapters devoted to orthogonal polynomials has increased from two to four; Meijer $G$-functions and Painlevé transcendents are now treated.

As we noted in the earlier book, the subject of special functions lacks a precise delineation, but it has a long and distinguished history. The remarks at the end of each chapter discuss the history, with numerous references and suggestions for further reading.

This book covers most of the standard topics and some that are less standard. We have tried to provide context for the discussion by emphasizing unifying ideas. The text and the problems provide proofs or proof outlines for nearly all the results and formulas.

We have also tried to keep the prerequisites to a minimum: a reasonable familiarity with power series and integrals, convergence, and the like. Some proofs rely on the basics of complex function theory, which are reviewed in the first appendix. Some familiarity with Hilbert space ideas, in the $L^2$ framework, is useful. The chapters on elliptic functions and on Painlevé transcendents rely more heavily than the rest of the book on concepts from complex analysis. The second appendix contains a quick development of basic results from Fourier analysis, including the Mellin transform.

The first chapter provides a general context for the discussion of the linear theory, especially in connection with special properties of the hypergeometric and confluent hypergeometric equations. Chapter 2 treats the gamma and beta functions at some length, with an introduction to the Riemann zeta function. Chapter 3 covers the relevant material from the theory of ordinary differential equations, including a characterization of the classical polynomials as eigenfunctions, and a discussion of separation of variables for equations involving the Laplacian.

The next four chapters are concerned with orthogonal polynomials on a real interval. Chapter 4 introduces the general theory, including three-term recurrence relations, Padé approximants, continued fractions, and Favard's theorem. The classical polynomials (Hermite, Laguerre, Jacobi) are treated in detail in Chapter 5, including asymptotic distribution of zeros. Chapter 6 introduces finite difference analogues of the classification theorem, yielding the classical discrete polynomials as well as neoclassical versions and the Askey scheme. Two methods of obtaining asymptotic results are presented in Chapter 7. In particular, the Riemann–Hilbert method is carried through for Hermite polynomials.

Chapters 8 through 11 contain a detailed treatment of the confluent hypergeometric equation, the hypergeometric equation, and special cases. These include Weber functions, Whittaker functions, Airy functions, cylinder functions (Bessel, Hankel, ...), spherical harmonics, and Legendre functions. Among the topics are linear relations, various transformations, integral representations, and asymptotics. Chapter 13 contains proofs of asymptotic results for these functions and for the classical polynomials.

In Chapter 12 we extend an earlier discussion of the special "recursive" property of the hypergeometric and confluent hypergeometric equations to equations of arbitrary order. This property characterizes the generalized hypergeometric equation. The corresponding solutions, the generalized hypergeometric functions, are covered in more detail than in the first version. Elliptic integrals, elliptic functions of Jacobi and Weierstrass, and theta functions are treated in Chapter 14.

The principal new topics, Meijer $G$-functions and Painlevé transcendents, have current theoretical and practical interest.

Meijer $G$-functions, which are special solutions of generalized hypergeometric equations, are introduced in Chapter 12. They generalize the classic Mellin–Barnes integral representations. The $G$-functions occur in probability and physics, and play a large role in compiling tables of integrals.

Chapter 15 has an extensive introduction to the classical and modern theory of Painlevé equations and their solutions, with emphasis on the second Painlevé equation, PII. Painlevé's method is introduced and PII is derived in detail. The isomonodromy method and Bäcklund transformations are introduced, and used to obtain rational solutions and information about general solutions. The Riemann–Hilbert method is used to derive a connection formula for solutions of PII(0). Applications include differential geometry, random matrix theory, integrable systems, and statistical physics.

The earlier book contained a concise summary of each chapter. These have been omitted here, partly to save space, and partly because the summaries often proved to be more annoying than helpful in use of the book for reference.

The first-named author acknowledges the efforts of some of his research collaborators, especially Peter Greiner, Bernard Gaveau, Yakar Kannai, David

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