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Orientation

The concept of a "special function" has no precise definition. From a practical point of view, a special function is a function of one variable that is (a) not one of the "elementary functions" – algebraic functions, trigonometric functions, the exponential, the logarithm, and functions constructed algebraically from these functions – and is (b) a function about which one can find information in many of the books about special functions. A large amount of such information has been accumulated over a period of three centuries. Like such elementary functions as the exponential and trigonometric functions, special functions come up in numerous contexts. These contexts include both pure mathematics and applications, ranging from number theory and combinatorics to probability and physical science.

The majority of special functions that are treated in many of the general books on the subject are solutions of certain second-order linear differential equations. Indeed, these functions were discovered through the study of physical problems: vibrations, heat flow, equilibrium, and so on. The associated equations are partial differential equations of second-order. In some coordinate systems these equations can be solved by separation of variables, leading to the second-order ordinary differential equations in question. (Solutions of the analogous *first-order* linear differential equations are elementary functions.)

Despite the long list of adjectives and proper names attached to this class of special functions (hypergeometric, confluent hypergeometric, cylinder, parabolic cylinder, spherical, Airy, Bessel, Hankel, Hermite, Kelvin, Kummer, Laguerre, Legendre, Macdonald, Neumann, Weber, Whittaker, ...), each of them is closely related to one of two families of equations: the confluent hypergeometric equation(s)

$$x\,u''(x) + (c - x)\,u'(x) - a\,u(x) \;=\; 0 \tag{1.0.1}$$

and the hypergeometric equation(s)

$$x(1-x)\,u''(x) + [c - (a+b+1)x]\,u'(x) - ab\,u(x) \;=\; 0. \tag{1.0.2}$$

The parameters a, b, c are real or complex constants.

Some solutions of these equations are polynomials: up to a linear change of variables, they are the "classical orthogonal polynomials." Again there are many names attached: Chebyshev, Gegenbauer, Hermite, Jacobi, Laguerre, Legendre, ultraspherical. In this introductory chapter we discuss one context in which these equations, and (up to normalization) no others, arise. We also shall see how two equations can, in principle, give rise to such a menagerie of functions.

Some special functions are *not* closely connected to linear differential equations. These exceptions include the gamma function, the beta function, elliptic functions, and the Painlevé transcendents.

The gamma and beta functions evaluate certain integrals. They are indispensable in many calculations, especially in connection with the class of functions mentioned earlier, as we illustrate below.

Elliptic functions arise as solutions of a simple *nonlinear* second-order differential equation, and also in connection with integrating certain algebraic functions. They have a wide range of applications, from number theory to integrable systems.

The Painlevé transcendents are solutions of a class of nonlinear second-order equations that share a crucial property with the equations that characterize elliptic functions, in that the solutions are single-valued in certain fixed domains, independent of the initial conditions.

1.1 Power series solutions

The general homogeneous linear second-order equation is

$$p(x)\,u''(x) + q(x)\,u'(x) + r(x)\,u(x) \;=\; 0, \tag{1.1.1}$$

with p not identically zero. We assume here that the coefficient functions p, q, and r are holomorphic (analytic) in a neighborhood of the origin.

If a function u is holomorphic in a neighborhood of the origin, then the function on the left side of (1.1.1) is also holomorphic in a neighborhood of the origin. The coefficients of the power series expansion of this function can be computed from the coefficients of the expansions of the functions p, q, r, and u. Under these assumptions, (1.1.1) is equivalent to the sequence of equations obtained by setting the coefficients of the expansion of the left side equal to zero. Specifically, suppose that the coefficient functions p, q, r have series expansions

$$p(x) = \sum_{k=0}^{\infty} p_k\, x^k, \quad q(x) = \sum_{k=0}^{\infty} q_k\, x^k, \quad r(x) = \sum_{k=0}^{\infty} r_k\, x^k,$$

and u has the expansion

$$u(x) = \sum_{k=0}^{\infty} u_k x^k.$$

Then the constant term and the coefficients of x and x^2 on the left side of (1.1.1) are

$$2p_0u_2 + q_0u_1 + r_0u_0, \tag{1.1.2}$$
$$6p_0u_3 + 2p_1u_2 + 2q_0u_2 + q_1u_1 + r_1u_0 + r_0u_1,$$
$$12p_0u_4 + 6p_1u_3 + 2p_2u_2 + 3q_0u_3 + 2q_1u_2 + q_2u_1 + r_0u_2 + r_1u_1 + r_2u_0,$$

respectively. The sequence of equations equivalent to (1.1.1) is the sequence

$$\sum_{j+k=n,k\geq 0} (k+2)(k+1)p_j u_{k+2} + \sum_{j+k=n,k\geq 0} (k+1)q_j u_{k+1} + \sum_{j+k=n,k\geq 0} r_j u_k = 0, \qquad n = 0,1,2,\ldots \tag{1.1.3}$$

We say that (1.1.1) is *recursive* if it has a nonzero solution u that is holomorphic in a neighborhood of the origin, and the equations (1.1.3) determine the coefficients $\{u_n\}$ by a simple recursion: the nth equation determines u_n in terms of u_{n-1} alone. Suppose that (1.1.1) is recursive. Then the first of the equations (1.1.2) should involve u_1 but not u_2, so $p_0 = 0$, $q_0 \neq 0$. The second equation should not involve u_3 or u_0, so $r_1 = 0$. Similarly, the third equation shows that $q_2 = r_2 = 0$. Continuing, we obtain

$$p_0 = 0, \quad p_j = 0,\ j \geq 3; \quad q_j = 0,\ j \geq 2; \quad r_j = 0,\ j \geq 1.$$

Collecting terms, we see that the nth equation is

$$[(n+1)np_1 + (n+1)q_0]\, u_{n+1} + [n(n-1)p_2 + nq_1 + r_0]\, u_n = 0.$$

For special values of the parameters p_1, p_2, q_0, q_1, r_0, one of these coefficients may vanish for some value of n. In such a case, either the recursion breaks down, or the solution u is a polynomial. We assume that this does not happen. Thus

$$u_{n+1} = -\frac{n(n-1)p_2 + nq_1 + r_0}{(n+1)np_1 + (n+1)q_0}\, u_n. \tag{1.1.4}$$

Assume $u_0 \neq 0$. If $p_1 = 0$ but $p_2 \neq 0$, the series $\sum_{n=0}^{\infty} u_n x^n$ diverges for all $x \neq 0$ (ratio test). Therefore, up to normalization – a linear change of coordinates and a multiplicative constant – we may assume that $p(x)$ has one of the two forms $p(x) = x(1-x)$ or $p(x) = x$.

If $p(x) = x(1-x)$, then (1.1.1) has the form

$$x(1-x)\, u''(x) + (q_0 + q_1 x)\, u'(x) + r_0\, u(x) = 0.$$

Constants a and b can be chosen so that this becomes (1.0.2).

If $p(x) = x$ and $q_1 \neq 0$ we may replace x by a multiple of x and take $q_1 = -1$. Then (1.1.1) has the form (1.0.1).

Finally, suppose $p(x) = x$ and $q_1 = 0$. If also $r_0 = 0$, then (1.1.1) is a first-order equation for u'. Otherwise, we may replace x by a multiple of x and take $r_0 = 1$. Then (1.1.1) has the form

$$x\,u''(x) + c\,u'(x) + u(x) \;=\; 0. \tag{1.1.5}$$

This equation is not obviously related to either (1.0.1) or (1.0.2). However, it can be shown that it becomes a special case of (1.0.1) after a change of variable and a "gauge transformation" (see Exercise 5).

In summary: up to certain normalizations, an equation of the form (1.1.1) is recursive if and only if it has one of the three forms (1.0.1), (1.0.2), or (1.1.5). Moreover, (1.1.5) can be transformed to a case of (1.0.1).

Let us note briefly the answer to the analogous question for a homogeneous linear *first-order* equation

$$q(x)\,u'(x) + r(x)\,u(x) \;=\; 0, \tag{1.1.6}$$

with q not identically zero. This amounts to taking $p = 0$ in the argument above. The conclusion is again that q is a polynomial of degree at most one, with $q_0 \neq 0$, while $r = r_0$ is constant. Up to normalization, $q(x)$ has one of the two forms $q(x) = 1$ or $q(x) = x - 1$. Thus the equation has one of the two forms

$$u'(x) - a\,u(x) \;=\; 0; \qquad (x-1)u'(x) - a\,u(x) \;=\; 0,$$

with solutions

$$u(x) \;=\; c\,e^{ax}, \quad u(x) \;=\; c\,(x-1)^a,$$

respectively.

The analogous question for homogeneous linear equations of *arbitrary* order is taken up in Chapter 12, Section 12.2.

Let us return to the confluent hypergeometric equation (1.0.1). The power series solution with $u_0 = 1$ is sometimes denoted $M(a,c;x)$. It can be calculated easily from the recursion (1.1.4). The result is

$$M(a,c;x) \;=\; \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n\, n!} x^n, \quad c \neq 0, -1, -2, \ldots \tag{1.1.7}$$

Here the "shifted factorial" or "Pochhammer symbol" $(a)_n$ is defined by

$$(a)_0 = 1, \qquad (a)_n = a(a+1)(a+2)\cdots(a+n-1), \tag{1.1.8}$$

so that $(1)_n = n!$. The series (1.1.7) converges for all complex x (ratio test), so M is an entire function of x.

The special nature of (1.0.1) is reflected in the special nature of the coefficients of M. It leads to a number of relationships among these functions when the parameters (a,b) are varied. For example, a comparison of coefficients shows that the three "contiguous" functions $M(a,c;x)$, $M(a+1,c;x)$, and $M(a,c-1;x)$ are related by

$$(a-c+1)\,M(a,c;x) - a\,M(a+1,c;x) + (c-1)\,M(a,c-1;x) = 0. \qquad (1.1.9)$$

Similar relations hold whenever the respective parameters differ by integers.

Equations (1.0.2) and (1.1.1) have solutions with expansions similar to (1.1.7), as do the generalizations considered in Chapter 12.

1.2 The gamma and beta functions

The gamma function

$$\Gamma(a) = \int_0^\infty e^{-t}\,t^{a-1}\,dt, \qquad \mathrm{Re}\,a > 0,$$

satisfies the functional equation $a\,\Gamma(a) = \Gamma(a+1)$. More generally, the shifted factorial (1.1.8) can be written as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

It is sometimes convenient to use this form in series like (1.1.7).

A related function is the beta function, or beta integral,

$$\mathrm{B}(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1}\,ds, \qquad \mathrm{Re}\,a > 0,\ \mathrm{Re}\,b > 0,$$

which can be evaluated in terms of the gamma function:

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)};$$

see the next chapter. These identities can be used to obtain a representation of the function M in (1.1.7) as an integral, when $\mathrm{Re}\,c > \mathrm{Re}\,a > 0$. In fact,

$$\begin{aligned}\frac{(a)_n}{(c)_n} &= \frac{\Gamma(a+n)}{\Gamma(a)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)}\\ &= \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\,\mathrm{B}(a+n,c-a)\\ &= \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_0^1 s^{n+a-1}(1-s)^{c-a-1}\,ds.\end{aligned}$$

Therefore

$$M(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 \left\{ s^{a-1}(1-s)^{c-a-1} \sum_{n=0}^{\infty} \frac{(sx)^n}{n!} \right\} ds$$

$$= \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 s^{a-1}(1-s)^{c-a-1} e^{sx}\, ds. \tag{1.2.1}$$

This integral representation is useful in obtaining information that is not evident from the power series expansion (1.1.7). Other integral representations occur naturally in the context of the Meijer G-functions in Chapter 12.

1.3 Three questions

First question: *How can it be that so many of the functions mentioned in the introduction to this chapter can be associated with just two equations,* (1.0.1) *and* (1.0.2)*?*

Part of the answer is that different solutions of the same equation may have different names. An elementary example is the equation

$$u''(x) - u(x) \;=\; 0. \tag{1.3.1}$$

One might wish to normalize a solution by imposing a condition at the origin like

$$u(0) \;=\; 0 \quad \text{or} \quad u'(0) \;=\; 0,$$

leading to $u(x) = \sinh x$ or $u(x) = \cosh x$, respectively, or a condition at infinity like

$$\lim_{x\to-\infty} u(x) \;=\; 0 \quad \text{or} \quad \lim_{x\to+\infty} u(x) \;=\; 0,$$

leading to $u(x) = e^x$ or $u(x) = e^{-x}$, respectively. Similarly, Bessel functions, Neumann functions, and both kinds of Hankel functions are four solutions of a single equation, distinguished by conditions at the origin or at infinity.

The rest of the answer to the question is that one can transform solutions of one second-order linear differential equation into solutions of another, in two simple ways. One such transformation is a change of variables. For example, starting with the equation

$$u''(x) - 2x\,u'(x) + \lambda\, u(x) \;=\; 0, \tag{1.3.2}$$

suppose $u(x) = v(x^2)$. It is not difficult to show that (1.3.2) is equivalent to the equation

$$y\,v''(y) + \left(\frac{1}{2} - y\right) v'(y) + \frac{1}{4}\lambda\, v(y) \;=\; 0,$$

which is the case $a = -\frac{1}{4}\lambda$, $c = \frac{1}{2}$ of (1.0.1). Therefore even solutions of (1.3.2) can be identified with certain solutions of (1.0.1). The same is true of odd solutions: see Exercise 12. An even simpler example is the change $u(x) = v(ix)$ in (1.3.1), leading to $v'' + v = 0$, and the trigonometric and complex exponential solutions $\sin x$, $\cos x$, e^{ix}, e^{-ix}.

The second type of transformation is a "gauge transformation." For example, if the function u in (1.3.2) is written in the form

$$u(x) \; = \; e^{x^2/2}\, v(x),$$

then (1.3.2) is equivalent to an equation with no first-order term:

$$v''(x) + (1 + \lambda - x^2)\, v(x) \; = \; 0. \tag{1.3.3}$$

Each of the functions mentioned in the third paragraph of the introduction to this chapter is a solution of an equation that can be obtained from (1.0.1) or (1.0.2) by one or both of a change of variable and a gauge transformation.

Second question: *What does one want to know about these functions?*

As we noted above, solutions of an equation of the form (1.1.1) can be chosen uniquely through various normalizations, such as behavior as $x \to 0$ or as $x \to \infty$. The solution (1.1.7) of (1.0.1) is normalized by the condition $u(0) = 1$. Having explicit formulas, like (1.1.7) for the function M, can be very useful. On the other hand, understanding the behavior as $x \to +\infty$ is not always straightforward. The integral representation (1.2.1) allows one to compute this behavior for M (see Exercise 18). This example illustrates why it can be useful to have an integral representation (with an integrand that is well understood).

Any three solutions of a second-order linear equation (1.1.1) satisfy a linear relationship, and one wants to compute the coefficients of such a relationship. An important tool in this and in other aspects of the theory is the computation of the Wronskian of two solutions u_1, u_2:

$$W(u_1, u_2)(x) \; \equiv \; u_1(x)\, u_2'(x) - u_2(x)\, u_1'(x).$$

In particular, these two solutions are linearly independent if and only if the Wronskian does not vanish.

Because of the special nature of (1.0.1) and (1.0.2) and the equations derived from them, solutions satisfy various linear relationships like (1.1.9). One wants to determine a set of relationships that generate all such relationships.

Finally, the coefficient of the zero-order term in equations like (1.0.1), (1.0.2), and (1.3.3) is an important parameter, and one often wants to know how a given normalized solution like $M(a, c; x)$ varies as the parameter approaches $\pm\infty$. In (1.3.3), denote by v_λ the even solution normalized by $v_\lambda(0) = 1$. As

$1+\lambda=\mu^2 \to +\infty$, over any bounded interval the equation looks like a small perturbation of the equation $v''+\mu^2 v=0$. Therefore it is plausible that

$$v_\lambda(x) \sim A_\lambda(x)\cos(\mu x+B_\lambda) \quad \text{as} \quad \lambda \to +\infty,$$

with $A_\lambda(x) > 0$. We want to compute the "amplitude function" $A_\lambda(x)$ and the "phase constant" B_λ. A word about notation like this in the preceding equation: the meaning of the statement

$$f(x) \sim A\,g(x) \quad \text{as} \quad x \to \infty$$

is

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = A, \quad A \neq 0.$$

This is in slight conflict with the notation for an *asymptotic series expansion*:

$$f(x) \sim g(x)\sum_{n=0}^{\infty} a_n x^{-n} \quad \text{as} \quad x \to \infty.$$

This means that for every positive integer N, truncating the series at $n=N$ gives an approximation to order x^{-N-1}:

$$\frac{f(x)}{g(x)} - \sum_{n=0}^{N} a_n x^{-n} = O(x^{-N-1}) \quad \text{as} \quad x \to \infty.$$

As usual, the "big O" notation

$$h(x) = O(k(x)) \quad \text{as} \quad x \to \infty$$

means that there are constants A, B such that

$$\left|\frac{h(x)}{k(x)}\right| \le A \quad \text{if} \quad x \ge B.$$

The similar "small o" notation

$$h(x) = o(k(x))$$

means that

$$\lim_{x\to\infty} \frac{h(x)}{k(x)} = 0.$$

Third question: *Is this list of functions or related equations exhaustive, in any sense?*

A partial answer has been given: the requirement that the equation be "recursive" leads to just three cases, (1.0.1), (1.0.2), and (1.1.5), and the third of these three equations reduces to a case of the first equation. Two other answers are given in Chapter 3.

The first of the two answers in Chapter 3 starts with a question of mathematics: given that a differential operator of the form that occurs in (1.1.1),

$$p(x)\,\frac{d^2}{dx^2} + q(x)\,\frac{d}{dx} + r(x),$$

is self-adjoint with respect to a weight function on a (bounded or infinite) interval, under what circumstances will the eigenfunctions be polynomials? An example is the operator in (1.3.2), which is self-adjoint with respect to the weight function $w(x) = e^{-x^2}$ on the line:

$$\int_{-\infty}^{\infty} \left[u''(x) - 2x\,u'(x)\right] v(x)\,e^{-x^2}\,dx \;=\; \int_{-\infty}^{\infty} u(x)\left[v''(x) - 2x\,v'(x)\right] e^{-x^2}\,dx.$$

The eigenvalues are $\lambda = 2, 4, 6, \ldots$, in (1.3.2) and the Hermite polynomials are eigenfunctions. Up to normalization, the equation associated with such an operator is one of the three equations (1.0.1), (1.0.2) (after a simple change of variables), or (1.3.2). Moreover, as suggested above, (1.3.2) can be converted to two cases of (1.0.1).

(One can ask the same question in connection with *difference equations*, with the derivative d/dx replaced by difference operators:

$$\Delta_+ u(m) \;=\; u(m+1) - u(m), \qquad \Delta_- u(m) \;=\; u(m) - u(m-1).$$

The polynomials that arise in this way are the classical discrete orthogonal polynomials of Charlier, Krawtchouk, Meixner, and Chebyshev–Hahn. Taking differences in the complex direction instead leads to the remaining "semi-classical" orthogonal polynomials.)

The second of the two answers in Chapter 3 starts with a question of mathematical physics: given the Laplace equation

$$\Delta u(\mathbf{x}) \;=\; 0$$

or the Helmholtz equation

$$\Delta u(\mathbf{x}) + \lambda\,u(\mathbf{x}) \;=\; 0,$$

say in three variables, $\mathbf{x} = (x_1, x_2, x_3)$, what equations arise by separating variables in various coordinate systems (Cartesian, cylindrical, spherical, parabolic cylindrical)? Each of the equations so obtained can be related to either (1.0.1) or (1.0.2) by a gauge transformation and/or a change of variables.

1.4 Other special functions

The remaining special functions to be discussed in this book are also associated with differential equations. Generalized hypergeometric functions and Meijer G-functions are solutions of linear equations of any order that have the recursive property discussed in Section 1.1. Elliptic functions and Painlevé transcendents are solutions of special second-order equations that are not linear.

One of the simplest nonlinear second-order differential equations of mathematical physics is the equation that describes the motion of an ideal pendulum, which can be normalized to

$$2\theta''(t) = -\sin\theta(t). \tag{1.4.1}$$

Multiplying equation (1.4.1) by $\theta'(t)$ and integrating gives

$$\left[\theta'(t)\right]^2 = a + \cos\theta(t) \tag{1.4.2}$$

for some constant a. Let $u = \sin\frac{1}{2}\theta$. Then (1.4.2) takes the form

$$\left[u'(t)\right]^2 = A\left[1 - u(t)^2\right]\left[1 - k^2u(t)^2\right]. \tag{1.4.3}$$

By rescaling time t, we may take the constant A to be 1. Solving for t as a function of u leads to the integral form

$$t = \int_{u_0}^{u} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{1.4.4}$$

This is an instance of an elliptic integral – an integral of the form

$$\int_{u_0}^{u} R\left(x, \sqrt{P(x)}\right)\, dx, \tag{1.4.5}$$

where P is a polynomial of degree 3 or 4 with no repeated roots, and R is a rational function (quotient of polynomials) in two variables. If P had degree 2, then (1.4.5) could be integrated by a trigonometric substitution. For example,

$$\int_0^u \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} u;$$

equivalently,

$$t = \int_0^{\sin t} \frac{dx}{\sqrt{1-x^2}}.$$

Elliptic functions are the analogues, for the case where P has degree 3 or 4, of the trigonometric functions in the case of degree 2.

All second-order equations discussed up to this point have the form

$$u'' = \frac{P(x,u,u')}{Q(x,u,u')},$$