1 Basic Equations

1.1 Introduction

This chapter provides a brief review of unsteady gas dynamics and properties of shock and detonation waves. The medium of propagation of shocks and detonations is considered to be a perfect gas. The thermodynamics of perfect gases are first reviewed. The conservation equations in various variables (i.e., p, ρ, u, c, etc.) and in different coordinates systems, for example, Eulerian, Characteristic, Lagrangian, etc., are thereafter discussed. A brief discussion of waves of small amplitude (acoustic) and finite amplitude is given. Wave propagation is the foundation of non-steady compressible flows. The conservation equations across normal shock waves and the Rankine–Hugoniot relationships are then presented. This chapter is meant to provide the background to facilitate the reading of subsequent chapters.

1.2 Thermodynamics

We shall consider a perfect gas throughout this book. The equation of state for an ideal gas is given by

$$pv = RT, \quad \text{or} \quad p = \rho RT, \tag{1.2.1}$$

where $v = \frac{1}{\rho}$ is the specific volume, and $R = \frac{R_u}{M}$ is the gas constant, $R_u = 8.314\ \text{J/mol·K}$ is the universal gas constant and M is the molecular weight. The relationship between the energy functions (internal energy e and enthalpy $h = e + pv$) and the state variables is known as the caloric equation of state. For an ideal gas, e and h are functions only of the temperature, and we write

$$e = c_v T, \quad h = c_p T, \tag{1.2.2}$$

where c_v and c_p are functions of the temperature only. The relationship between c_p and c_v is

$$c_p - c_v = R. \tag{1.2.3}$$

The ratio of the specific heats is denoted by γ, that is,

$$\gamma = \frac{c_p}{c_v}. \tag{1.2.4}$$

In terms of p and ρ, Eq 1.2.2 can be written as

$$\begin{aligned} e = c_v T &= \frac{RT}{\gamma - 1} = \frac{pv}{\gamma - 1} = \frac{p}{\rho(\gamma - 1)}, \\ h = c_p T &= \frac{\gamma RT}{\gamma - 1} = \frac{\gamma pv}{\gamma - 1} = \frac{\gamma p}{\rho(\gamma - 1)}. \end{aligned} \tag{1.2.5}$$

The speed of sound is defined as

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s,$$

and for a perfect gas for which γ is constant, we write

$$c^2 = \gamma RT = \frac{\gamma p}{\rho}. \tag{1.2.6}$$

The entropy function, s, can be obtained from the relationship

$$T\mathrm{d}s = \mathrm{d}e + p\mathrm{d}v = \mathrm{d}h - v\mathrm{d}p. \tag{1.2.7}$$

For a perfect gas with constant γ, the above equations give,

$$\frac{s_2 - s_1}{R} = \ln\left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}}\left(\frac{v_2}{v_1}\right) = \ln\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{p_2}{p_1}\right). \tag{1.2.8}$$

For an isentropic process where $\mathrm{d}s = 0$, Eq 1.2.8 yields

$$\begin{aligned} T_2 v_2^{\gamma-1} &= T_1 v_1^{\gamma-1} = \frac{T}{\rho^{\gamma-1}} = \text{constant}. \\ \frac{T_2}{p_2^{\frac{\gamma-1}{\gamma}}} &= \frac{T_1}{p_1^{\frac{\gamma-1}{\gamma}}} = \frac{T}{p^{\frac{\gamma-1}{\gamma}}} = \text{constant}. \end{aligned} \tag{1.2.9}$$

Using the equation of state to eliminate T, we write

$$pv^{\gamma} = \frac{p}{\rho^{\gamma}} = \text{constant}. \tag{1.2.10}$$

The speed of sound can often be used to represent the energy function e and h. From Eqs 1.2.5 and 1.2.6, we obtain

$$e = \frac{c^2}{\gamma(\gamma - 1)} \qquad h = \frac{c^2}{\gamma - 1}. \tag{1.2.11}$$

1.3 Conservation Equations

We shall consider an inviscid, non-heat-conducting perfect gas. The conservation equations (Euler's equations) are given as

$$\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{u} = 0, \tag{1.3.1}$$

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho} \vec{\nabla} p = 0, \tag{1.3.2}$$

$$T\frac{Ds}{Dt} = \frac{De}{Dt} + p\frac{D\frac{1}{\rho}}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = 0, \tag{1.3.3}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}.$$

Equation 1.3.3 determines the change in entropy of a fluid particle as we follow its motion. When every particle has the same entropy, then $s =$ constant throughout. This is generally referred to as "homentropic" flow.

Using Eq 1.3.1 to eliminate $\frac{D\rho}{Dt}$ in Eq 1.3.3, we obtain

$$\frac{De}{Dt} + \frac{p}{\rho} \vec{\nabla} \cdot \vec{u} = 0. \tag{1.3.4}$$

Expressing the internal energy in terms of the sound speed (Eq 1.2.11), Eq 1.3.4 can be written as

$$\frac{Dc}{Dt} + \left(\frac{\gamma - 1}{2}\right) c \vec{\nabla} \cdot \vec{u} = 0. \tag{1.3.5}$$

The above equation is valid for particle isentropic flow. If this condition is not met, Eqs 1.3.3 and 1.3.5 give

$$\frac{Dc}{Dt} + \left(\frac{\gamma - 1}{2}\right) c \vec{\nabla} \cdot \vec{u} = \left(\frac{\gamma - 1}{2}\right) \frac{c}{R}\frac{Ds}{Dt}, \tag{1.3.6}$$

which is valid for non-isentropic flow. Using Eq 1.3.6, the momentum equation (Eq 1.3.4) can be written as

$$\frac{D\vec{u}}{Dt} + \left(\frac{2c}{\gamma - 1}\right) \vec{\nabla} c = T\vec{\nabla} s = \frac{c^2}{\gamma R} \vec{\nabla} s. \tag{1.3.7}$$

For particle isentropic flow or when there is no entropy gradient, where $\vec{\nabla} s = 0$, the right hand side of Eqs 1.3.6 and 1.3.7 vanishes.

Most of the time, we shall be considering one-dimensional non-steady flow with planar, cylindrical, and spherical symmetries. Equations 1.3.1 and 1.3.2 then become

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{j\rho u}{r} = 0, \tag{1.3.8}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{1.3.9}$$

where $j = 0, 1, 2$ for planar, cylindrical, and spherical geometries, respectively. Similarly, we may write Eqs 1.3.6 and 1.3.7 as

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial r} + \left(\frac{\gamma-1}{2}\right)c\frac{\partial u}{\partial r} + \left(\frac{\gamma-1}{2}\right)\frac{jcu}{r} = \left(\frac{\gamma-1}{2}\right)\frac{c}{R}\frac{\mathrm{D}s}{\mathrm{D}t}, \tag{1.3.10}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{2c}{\gamma-1}\frac{\partial c}{\partial r} = \frac{c^2}{\gamma R}\frac{\partial s}{\partial r}. \tag{1.3.11}$$

For particle isentropic flow, the right hand side of Eq 1.3.10 vanishes. If the entropy also is uniform, that is, $\frac{\partial s}{\partial r} = 0$, then the right hand side of Eq 1.3.11 also vanishes. Thus for homentropic flow, we write

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial r} + \left(\frac{\gamma-1}{2}\right)c\frac{\partial u}{\partial r} + \left(\frac{\gamma-1}{2}\right)\frac{jcu}{r} = 0, \tag{1.3.12}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{2c}{\gamma-1}\frac{\partial c}{\partial r} = 0. \tag{1.3.13}$$

Equations 1.3.12 and 1.3.13 give a pair of equations for the dependent variables u and c. With c known, the pressure and density can be obtained from the isentropic relationships.

Equations derived previously (Eqs 1.3.8–1.3.13) are based on the Eulerian description where the flow variables are specified as a function of the position and time, that is, $p(r,t)$, $\rho(r,t)$, etc. An alternate description is the Lagrangian description where the fluid state of a particle is specified as a function of time. The Lagrangian variable, a, is generally chosen to be the initial position of the fluid particle. At a later time, t, the position is given by $r(a,t)$. The conservation equations for the general case of two- or three-dimensional flows in Lagrangian coordinates are quite complex. We shall consider only one-dimensional flow here. For simplicity, we shall derive the equations for spherical symmetry first and then generalize them for other geometries afterwards.

Let the initial position of a particular fluid particle at $t = 0$ be a. At a later time t, the position of the particle will be $r(a,t)$. Thus the mass in the spherical shell, $\rho_0 4\pi a^2 da$, at $t = 0$ will be $\rho 4\pi r^2 dr$ at a later time t. The conservation of mass is then given by

$$\rho_0 4\pi a^2 \mathrm{d}a = \rho 4\pi r^2 \mathrm{d}r(a,t). \tag{1.3.14}$$

Since

$$\mathrm{d}r(a,t) = \frac{\partial r}{\partial t}\mathrm{d}t + \frac{\partial r}{\partial a}\mathrm{d}a,$$

at constant time t, $\frac{dr}{da} = (\frac{\partial r}{\partial a})_t$, and the conservation of mass can be written as

$$\left(\frac{\partial r}{\partial a}\right)_t = \frac{\rho_0}{\rho}\left(\frac{a}{r}\right)^2. \tag{1.3.15}$$

The conservation equation for momentum of the particle at time t, can be written as

$$(\rho 4\pi r^2 dr)\left(\frac{\partial u}{\partial t}\right)_a = -4\pi r^2 dp,$$

and using Eq 1.3.15, the above expression becomes

$$\left(\frac{\partial u}{\partial t}\right)_a = -\frac{1}{\rho_0}\left(\frac{r}{a}\right)^2\left(\frac{\partial p}{\partial a}\right)_t. \tag{1.3.16}$$

For particle isentropic flow, the entropy of a fluid particle remains constant with time. Thus

$$\left(\frac{\partial s}{\partial t}\right)_a = 0. \tag{1.3.17}$$

Generalizing to planar and cylindrical geometries, we write Eqs 1.3.15 and 1.3.16 as

$$\frac{\partial r}{\partial a} = \frac{\rho_0}{\rho}\left(\frac{a}{r}\right)^j, \tag{1.3.18}$$

and

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\left(\frac{a}{r}\right)^j\frac{\partial p}{\partial a}, \tag{1.3.19}$$

where $j = 0, 1, 2$ for planar, cylindrical, and spherical symmetries, respectively. It is understood that a is kept constant for the partial differentiation with respect to t and vice versa in Eqs 1.3.15.

1.4 Characteristic Equations

Euler equations are hyperbolic and have real characteristics. It is often convenient to integrate these equations along the characteristics. To obtain the conservation equations in characteristic form, we use the method of multipliers. Multiplying Eq 1.3.10 by α and Eq 1.3.11 by β and adding the resulting equations, we obtain after some rearrangements, the following:

$$\left[\frac{\partial}{\partial t} + \left(u + \frac{\beta}{\alpha}\left(\frac{2}{\gamma-1}\right)c\right)\frac{\partial}{\partial r}\right]c + \frac{\beta}{\alpha}\left[\frac{\partial}{\partial t} + \left(u + \frac{\alpha}{\beta}\left(\frac{\gamma-1}{2}\right)c\right)\frac{\partial}{\partial r}\right]u$$
$$= -\frac{\gamma-1}{2}\frac{jcu}{r} + \frac{\gamma-1}{2}\frac{c}{R}\frac{\mathrm{D}S}{\mathrm{D}t} + \frac{\beta}{\alpha}\frac{c^2}{\gamma R}\frac{\partial s}{\partial r}. \tag{1.4.1}$$

We wish to write the above equation as a total differential equation of the form

$$\frac{\mathrm{d}c}{\mathrm{d}t}+\frac{\beta}{\alpha}\frac{\mathrm{d}u}{\mathrm{d}t}=-\frac{\gamma-1}{2}\frac{jcu}{r}+\frac{\gamma-1}{2}\frac{c}{R}\frac{\mathrm{D}s}{\mathrm{D}t}+\frac{\beta}{\alpha}\frac{c^2}{\gamma R}\frac{\partial s}{\partial r}. \tag{1.4.2}$$

Since c and u are functions of r and t, we write

$$\frac{\mathrm{d}c}{\mathrm{d}t}=\frac{\partial c}{\partial t}+\frac{\partial c}{\partial r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right),$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right).$$

From Eq 1.4.1, we see that

$$\frac{\mathrm{d}r}{\mathrm{d}t}=u+\frac{\beta}{\alpha}\left(\frac{2}{\gamma-1}\right)c=u+\frac{\alpha}{\beta}\left(\frac{\gamma-1}{2}\right)c.$$

Solving for the multipliers, we get

$$\frac{\beta}{\alpha}=\pm\frac{\gamma-1}{2},$$

and Eq 1.4.1 can then be written as

$$\left[\frac{\partial}{\partial t}+(u\pm c)\frac{\partial}{\partial r}\right]c\pm\frac{\gamma-1}{2}\left[\frac{\partial}{\partial t}+(u\pm c)\frac{\partial}{\partial r}\right]u$$
$$=-\frac{\gamma-1}{2}\frac{jcu}{r}+\frac{\gamma-1}{2}\frac{c}{R}\frac{\mathrm{D}s}{\mathrm{D}t}\pm\frac{\gamma-1}{2}\frac{c^2}{\gamma R}\frac{\partial s}{\partial r}. \tag{1.4.3}$$

The characteristic curves are given by

$$\frac{\mathrm{d}r}{\mathrm{d}t}=u\pm c. \tag{1.4.4}$$

We may write Eq 1.4.3 alternately as

$$\frac{\delta}{\delta t}\left(\frac{2c}{\gamma-1}\pm u\right)=-\frac{jcu}{r}+\frac{c}{R}\frac{\mathrm{D}s}{\mathrm{D}t}\pm\frac{\gamma-1}{2}\frac{c^2}{\gamma R}\frac{\partial s}{\partial r}. \tag{1.4.5}$$

where

$$\frac{\delta}{\delta t}=\frac{\partial}{\partial t}+(u\pm c)\frac{\partial}{\partial r}.$$

Defining $J^{\pm}=\frac{2}{\gamma-1}c\pm u$, Eq 1.4.5 may be written as

$$\frac{\delta}{\delta t}J^{\pm}=-\frac{jcu}{r}+\frac{c}{R}\frac{\mathrm{D}s}{\mathrm{D}t}\pm\frac{c^2}{\gamma R}\frac{\partial s}{\partial r}. \tag{1.4.6}$$

For particle isentropic flow $\left(\frac{\mathrm{D}s}{\mathrm{D}t}=0\right)$ and if the entropy is also uniform, then $\frac{\partial s}{\partial r}=0$ and Eq 1.4.6 reduces to

$$\frac{\delta J^{\pm}}{\delta t}=-\frac{jcu}{r}. \tag{1.4.7}$$

For planar geometry where $j = 0$, Eq 1.4.7 can be integrated to yield

$$J^{\pm} = \text{constant} = \frac{2}{\gamma - 1} c \pm u.$$

If $c = c_0$ when $u = 0$, then

$$J^{\pm} = \frac{2}{\gamma - 1} c \pm u = \frac{2}{\gamma - 1} c_0,$$

and solving for c gives

$$c = c_0 \mp \frac{\gamma - 1}{2} u,$$

$$\text{or} \qquad u = \pm \frac{2}{\gamma - 1} (c_0 - c). \tag{1.4.8}$$

The above equations give $c(u)$ or $u(c)$, and we may also express c and u in terms of the Riemann invariants $J^{\pm}$, that is,

$$u = \left(\frac{J^+ - J^-}{2} \right),$$

$$c = \frac{\gamma - 1}{2} \left(\frac{J^+ + J^-}{2} \right). \tag{1.4.9}$$

Using the above equations we can write the characteristic equations (Eq 1.4.4) as

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u + c = \left(\frac{J^+ - J^-}{2} \right) + \frac{\gamma - 1}{2} \left(\frac{J^+ + J^-}{2} \right), \tag{1.4.10}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u - c = \left(\frac{J^+ - J^-}{2} \right) - \frac{\gamma - 1}{2} \left(\frac{J^+ + J^-}{2} \right). \tag{1.4.11}$$

Since $J^+ =$ constant along a C^+, $\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^+$ will depend on the value of J^-. Similarly for a C^-, the slope $\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^-$ will depend on the value of the J^+. For a uniform flow where J^+ and J^- are constants, $\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^{\pm} =$ constant and the characteristics become two sets of parallel lines.

For a non-steady flow adjacent to a uniform or stationary region, the Riemann invariant from the uniform region will be transmitted to the non-steady region via the characteristic. Thus one of the two Riemann invariants will be constant throughout both regions. Figure 1.1 shows the uniform and non-uniform regions. A J^- from the uniform region will be carried by a C^- characteristic into the non-steady region.

Thus $\frac{2}{\gamma-1} c - u$ is constant throughout and if $u = 0$, $c = c_0$ in the uniform region, that is, if

$$\frac{2}{\gamma - 1} c - u = \frac{2}{\gamma - 1} c_0,$$

$$\text{then} \qquad c = c_0 + \frac{\gamma - 1}{2} u. \tag{1.4.12}$$

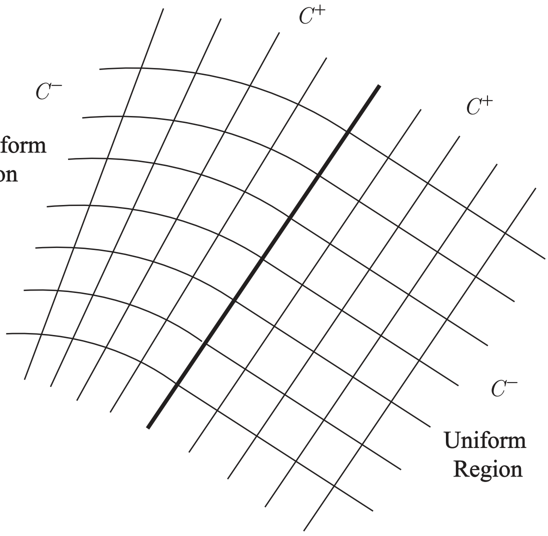

Figure 1.1 Uniform and Non-uniform Regions

Hence, a C^+ in the non-steady region will then be given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u + c = c_0 + \frac{\gamma + 1}{2}u,$$

which integrates to yield the trajectory of the C^+ as

$$r = \left(c_0 + \frac{\gamma + 1}{2}u\right)t + f(u), \tag{1.4.13}$$

where the constant of integration $f(u)$ can be determined from initial or boundary conditions. Along a C^+, $J^+ = \frac{2}{\gamma-1}c + u =$ constant and using Eq 1.3.12, $J^+ = \frac{2}{\gamma-1}(2c - c_0)$. Thus, each C^+ is characterized by a value of u (or alternately a value of c).

Similarly, if a C^+ carries a J^+ from a uniform region into the non-uniform region and the uniform region is defined by $u = 0$, $c = c_0$, then

$$J^+ = \frac{2}{\gamma - 1}c + u = \frac{2}{\gamma - 1}c_0 = \text{constant},$$

and thus

$$c = c_0 - \frac{\gamma - 1}{2}u. \tag{1.4.14}$$

A C^- in the non-uniform region is thus

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u - c = -c_0 + \frac{\gamma + 1}{2}u, \tag{1.4.15}$$

which integrates to yield

$$r = \left(-c_0 + \frac{\gamma+1}{2}u\right)t + f(u). \tag{1.4.16}$$

Combining Eqs 1.4.13 and 1.4.16, we write

$$r = \left(\pm c_0 + \frac{\gamma+1}{2}u\right)t + f(u), \tag{1.4.17}$$

for the trajectories of the $C^{\pm}$ in the non-uniform region adjacent to a uniform region where $u = 0$, $c = c_0$.

For the particular case where the characteristics originate from a point (e.g., $x = 0$ at $t = 0$), then $f(u) = 0$ and Eq 1.4.17 becomes

$$r = \left(\pm c_0 + \frac{\gamma+1}{2}u\right)t. \tag{1.4.18}$$

Solving for u, we get

$$\frac{u}{c_0} = \frac{2}{\gamma+1}\left(\frac{x}{c_0 t} \mp 1\right). \tag{1.4.19}$$

From Eqs 1.4.12 and 1.4.14, we get

$$\begin{aligned}\frac{c}{c_0} &= 1 \pm \frac{\gamma+1}{2}\frac{u}{c_0} = 1 \pm \left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{x}{c_0 t} \mp 1\right) \\ &= \frac{2}{\gamma+1}\left(1 \pm \frac{\gamma-1}{2}\frac{x}{c_0 t}\right).\end{aligned} \tag{1.4.20}$$

The pressure, density, and temperature can be obtained using the isentropic relationships and Eq 1.4.20, that is,

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{c}{c_0}\right)^{\frac{2\gamma}{\gamma-1}},$$

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\frac{1}{\gamma-1}}.$$

Equations 1.4.12, 1.4.14, and 1.4.17 describe what is referred to as a “simple wave” flow.

1.5 Acoustic Waves

Small flow perturbations propagate as acoustic waves governed by the linear wave equation. Writing $p = p_0 + p'$, $\rho = \rho_0 + \rho'$, $\vec{u} = \vec{u}'$ where $\frac{p'}{p_0} \ll 1$, $\frac{\rho'}{\rho_0} \ll 1$, and $\frac{u'}{c_0} \ll 1$, Eqs 1.3.1 and 1.3.2 reduce to the following equations when second-order terms are neglected,

$$\frac{\partial \rho'}{\partial t} + \vec{\nabla} \cdot \rho_0 \vec{u}' = 0, \quad (1.5.1)$$

$$\frac{\partial \vec{u}'}{\partial t} + \left(\vec{u}' \cdot \vec{\nabla}\right) \vec{u}' + \frac{1}{\rho_0} \vec{\nabla} p' = 0. \quad (1.5.2)$$

For low frequency perturbations, the convection term is small compared to the non-steady term, and Eq 1.5.2 becomes

$$\frac{\partial \vec{u}'}{\partial t} + \frac{1}{\rho_0} \vec{\nabla} p' = 0. \quad (1.5.3)$$

Since $c_0^2 = \frac{p'}{\rho'}$, Eq 1.5.1 can be written as

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \vec{\nabla} \cdot \vec{u}' = 0, \quad (1.5.4)$$

and taking the divergence of Eq 1.5.3 and combining with Eq 1.5.4, we get

$$\frac{\partial^2 p'}{\partial t^2} - c_0^2 \nabla^2 p' = 0, \quad (1.5.5)$$

which is the linear wave equation in the variable p'. Defining a velocity potential ϕ such that $\vec{\nabla} \phi = \vec{u}'$, we may write Eq 1.5.3 as

$$\vec{\nabla} \frac{\partial \phi}{\partial t} + \vec{\nabla} \frac{p'}{\rho_0} = 0.$$

Hence

$$p' = -\rho_0 \frac{\partial \phi}{\partial t}. \quad (1.5.6)$$

Differentiating the above equation with respect to time,

$$\frac{\partial p'}{\partial t} = -\rho_0 \frac{\partial^2 \phi}{\partial t^2},$$

and replacing $\frac{\partial p'}{\partial t}$ in Eq 1.5.4 by the above equation yields

$$\phi_{tt} - c_0^2 \nabla^2 \phi = 0, \quad (1.5.7)$$

which is the linear wave equation in the perturbation velocity potential, ϕ. For one-dimensional planar, cylindrical, and spherical waves, Eq 1.5.7 can be written as

$$\phi_{tt} - \frac{c_0^2}{r^j} \frac{\partial}{\partial r} \left(r^j \phi_r\right) = 0, \quad (1.5.8)$$

where $j = 0, 1, 2$ for the planar, cylindrical, and spherical geometries, respectively.

For the planar case $j = 0$, the solution of Eq 1.5.8 is given by

$$\phi = f_1(r - c_0 t) + f_2(r + c_0 t), \quad (1.5.9)$$