A Second Course in Linear Algebra

Linear algebra is a fundamental tool in many fields, including mathematics and statistics, computer science, economics, and the physical and biological sciences. This undergraduate textbook offers a complete second course in linear algebra, tailored to help students transition from basic theory to advanced topics and applications. Concise chapters promote a focused progression through essential ideas, and contain many examples and illustrative graphics. In addition, each chapter contains a bullet list summarizing important concepts, and the book includes over 600 exercises to aid the reader's understanding.

Topics are derived and discussed in detail, including the singular value decomposition, the Jordan canonical form, the spectral theorem, the *QR* factorization, normal matrices, Hermitian matrices (of interest to physics students), and positive definite matrices (of interest to statistics students).

Stephan Ramon Garcia is Associate Professor of Mathematics at Pomona College. He is the author of two books and over seventy research articles in operator theory, complex analysis, matrix analysis, number theory, discrete geometry, and other fields. He is on the editorial boards of the *Proceedings of the American Mathematical Society* (2016–), *Involve* (2011–), and *The American Mathematical Monthly* (2017–). He received three NSF research grants as principal investigator, five teaching awards from three different institutions, and was twice nominated by Pomona College for the CASE US Professors of the Year award.

Roger A. Horn was Research Professor of Mathematics at the University of Utah. His publications include the Cambridge University Press books Matrix Analysis and Topics in Matrix Analysis as well as more than 100 research articles in matrix analysis, statistics, health services research, complex variables, probability, differential geometry, and analytic number theory. He was Editor of *The American Mathematical Monthly* (1996–2001), the MAA *Spectrum* book series (1992–1995), and the MAA *Carus Mathematical Monographs* (2002–2005). He has also served on the editorial boards of the *SIAM Journal of Matrix Analysis*, *Linear Algebra and its Applications*, and the *Electronic Journal of Linear Algebra*.

A Second Course in Linear Algebra

Stephan Ramon Garcia
Pomona College, California

Roger A. Horn
University of Utah

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107103818

First published 2017

Printed in the United States of America by Sheridan Books, Inc.

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication data
Names: Garcia, Stephan Ramon. | Horn, Roger A.
Title: A second course in linear algebra / Stephan Ramon Garcia, Pomona College, California, Roger A. Horn, University of Utah.
Description: Cambridge : Cambridge University Press, 2017. | Includes bibliographical references and index.
Identifiers: LCCN 2016033754 | ISBN 9781107103818 (Hardback : alk. paper)
Subjects: LCSH: Algebras, Linear–Textbooks.
Classification: LCC QA184.2 .G37 2017 | DDC 512/.5–dc23
LC record available at https://lccn.loc.gov/2016033754

ISBN 978-1-107-10381-8 Hardback

To our families:

Gizem, Reyhan, and Altay
Susan;
Craig, Cori, Cole, and Carson;
Howard, Heidi, Archer, and Ella Ceres

Contents

Cambridge University Press
978-1-107-10381-8 — A Second Course in Linear Algebra
Stephan Ramon Garcia , Roger A. Horn
Frontmatter
More Information

Preface

Linear algebra and matrix methods are increasingly relevant in a world focused on the acquisition and analysis of data. Consequently, this book is intended for students of pure and applied mathematics, computer science, economics, engineering, mathematical biology, operations research, physics, and statistics. We assume that the reader has completed a lower-division calculus sequence and a first course in linear algebra.

Noteworthy features of this book include the following:

- Block matrices are employed systematically.
- Matrices and matrix factorizations are emphasized.
- Transformations that involve unitary matrices are emphasized because they are associated with feasible and stable algorithms.
- Numerous examples appear throughout the text.
- Figures illustrate the geometric foundations of linear algebra.
- Topics for a one-semester course are arranged in a sequence of short chapters.
- Many chapters conclude with sections devoted to special topics.
- Each chapter includes a problem section (more than 600 problems in total).
- Notes sections provide references to sources of additional information.
- Each chapter concludes with a bullet list of important concepts introduced in the chapter.
- Symbols used in the book are listed in a table of notation, with page references.
- An index with more than 1700 entries helps locate concepts and definitions, and enhances the utility of the book as a reference.

Matrices and vector spaces in the book are over the complex field. The use of complex scalars facilitates the study of eigenvalues and is consistent with modern numerical linear algebra software. Moreover, it is aligned with applications in physics (complex wave functions and Hermitian matrices in quantum mechanics), electrical engineering (analysis of circuits and signals in which both phase and amplitude are important), statistics (time series and characteristic functions), and computer science (fast Fourier transforms, convergent matrices in iterative algorithms, and quantum computing).

While studying linear algebra with this book, students can observe and practice good mathematical communication skills. These skills include how to state (and read) a theorem carefully; how to choose (and use) hypotheses; how to prove a statement by induction, by contradiction, or by proving its contrapositive; how to improve a theorem by weakening its hypotheses or strengthening its conclusions; how to use counterexamples; and how to write a cogent solution to a problem.

Many topics that are useful in applications of linear algebra fall outside the realm of linear transformations and similarity, so they may be absent from textbooks that adopt an abstract operator approach. These include:

- Geršgorin's theorem
- Householder matrices
- The *QR* factorization
- Block matrices
- Discrete Fourier transforms
- Circulant matrices
- Matrices with nonnegative entries (Markov matrices)
- The singular value and compact singular value decompositions
- Low-rank approximations to a data matrix
- Generalized inverses (Moore–Penrose inverses)
- Positive semidefinite matrices
- Hadamard (entrywise) and Kronecker (tensor) products
- Matrix norms
- Least squares and minimum norm solutions
- Complex symmetric matrices
- Inertia of normal matrices
- Eigenvalue and singular value interlacing
- Inequalities involving eigenvalues, singular values, and diagonal entries

The book is organized as follows:

Chapter 0 is a review of definitions and results from elementary linear algebra.

Chapters 1 and 2 review complex and real vector spaces, including linear independence, bases, dimension, rank, and matrix representations of linear transformations.

The "second course" topics begin in Chapter 3, which establishes the block-matrix paradigm used throughout the book.

Chapters 4 and 5 review geometry in the Euclidean plane and use it to motivate axioms for inner product and normed linear spaces. Topics include orthogonal vectors, orthogonal projections, orthonormal bases, orthogonalization, the Riesz representation theorem, adjoints, and applications of the theory to Fourier series.

Chapter 6 introduces unitary matrices, which are used in constructions throughout the rest of the book. Householder matrices are used to construct the *QR* factorization, which is employed in many numerical algorithms.

Chapter 7 discusses orthogonal projections, best approximations, least squares/minimum norm solutions of linear systems, and use of the QR factorization to solve the normal equations.

Chapter 8 introduces eigenvalues, eigenvectors, and geometric multiplicity. We show that an $n \times n$ complex matrix has between one and n distinct eigenvalues, and use Geršgorin's theorem to identify a region in the complex plane that contains them.

Chapter 9 deals with the characteristic polynomial and algebraic multiplicity. We develop criteria for diagonalizability and define primary matrix functions of a diagonalizable matrix. Topics include Fibonacci numbers, the eigenvalues of AB and BA, commutants, and simultaneous diagonalization.

Chapter 10 contains Schur's remarkable theorem that every square matrix is unitarily similar to an upper triangular matrix (with a related result for a commuting family). Schur's theorem is used to show that every square matrix is annihilated by its characteristic polynomial. The latter result motivates introduction of the minimal polynomial and a study of its properties. Sylvester's theorem on linear matrix equations is proved and used to show that every square matrix is similar to a block diagonal matrix with unispectral diagonal blocks.

Chapter 11 builds on the preceding chapter to show that every square matrix is similar to a special block diagonal upper bidiagonal matrix (its Jordan canonical form) that is unique up to permutation of its direct summands. Applications of the Jordan canonical form include initial value problems for linear systems of differential equations, an analysis of the Jordan structures of AB and BA, characterizations of convergent and power-bounded matrices, and a limit theorem for Markov matrices that have positive entries.

Chapter 12 is about normal matrices: matrices that commute with their conjugate transpose. The spectral theorem says that a matrix is normal if and only if it is unitarily diagonalizable; many other equivalent characterizations are known. Hermitian, skew-Hermitian, unitary, real orthogonal, real symmetric, and circulant matrices are all normal.

Positive semidefinite matrices are the subject of Chapter 13. These matrices arise in statistics (correlation matrices and the normal equations), mechanics (kinetic and potential energy in a vibrating system), and geometry (ellipsoids). Topics include the square root function, Cholesky factorization, and the Hadamard and Kronecker products.

The principal result in Chapter 14 is the singular value decomposition, which is at the heart of many modern numerical algorithms in statistics, control theory, approximation, image compression, and data analysis. Topics include the compact singular value decomposition and polar decompositions, with special attention to uniqueness of these factorizations.

In Chapter 15 the singular value decomposition is used to compress an image or data matrix. Other applications of the singular value decomposition discussed are the generalized inverse (Moore–Penrose inverse) of a matrix; inequalities between singular values and eigenvalues; the spectral norm of a matrix; complex symmetric matrices; and idempotent matrices.

Chapter 16 investigates eigenvalue interlacing phenomena for Hermitian matrices that are bordered or are subjected to an additive perturbation. Related results include an interlacing theorem for singular values, a determinant criterion for positive definiteness, and inequalities that characterize eigenvalues and diagonal entries of a Hermitian matrix. We prove Sylvester's inertia theorem for Hermitian matrices and a generalized inertia theorem for normal matrices.

A comprehensive list of symbols and notation (with page references) follows the Preface. A review of complex numbers and a list of references follow Chapter 16. A detailed index is at the end of the book.

The cover art is an image of a 2002 oil painting “Summer Again” (72 × 52 inches) by Lun-Yi Tsai, a New York City artist whose work has often been inspired by mathematical themes.

We thank Zachary Glassman for producing many of our illustrations and for answering our LaTeX questions.

We thank Dennis Merino, Russ Merris, and Zhongshan Li for carefully reading a preliminary draft.

We thank the students who attended the first author’s advanced linear algebra courses at Pomona College during Fall 2014 and Fall 2015. In particular, we thank Ahmed Al Fares, Andreas Biekert, Andi Chen, Wanning Chen, Alex Cloud, Bill DeRose, Jacob Fiksel, Logan Gilbert, Sheridan Grant, Adam He, David Khatami, Cheng Wai Koo, Bo Li, Shiyue Li, Samantha Morrison, Nathanael Roy, Michael Someck, Sallie Walecka, and Wentao Yuan for catching several mistakes in a preliminary version of this text.

Special thanks to Ciaran Evans, Elizabeth Sarapata, Adam Starr, and Adam Waterbury for their eagle-eyed reading of the text.

S.R.G.
R.A.H.

Notation

$\in, \notin$	is / is not an element of
$\subseteq$	is a subset of
$\varnothing$	the empty set
$\times$	Cartesian product
$f : X \to Y$	f is a function from X into Y
$\Longrightarrow$	implies
$\Longleftrightarrow$	if and only if
$x \mapsto y$	implicit definition of a function that maps x to y
$\mathbb{N} = \{1, 2, 3, \ldots\}$	the set of all natural numbers
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	the set of all integers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{F}$	field of scalars ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)
$[a, b]$	a real interval that includes its endpoints a, b
$\mathcal{U}, \mathcal{V}, \mathcal{W}$	vector spaces
$\mathscr{U}, \mathscr{V}$	subsets of vector spaces
$a, b, c, \ldots$	scalars
$\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$	(column) vectors
$A, B, C, \ldots$	matrices
δ_{ij}	Kronecker delta (p. 3)
I_n	$n \times n$ identity matrix (p. 3)
I	identity matrix (size inferred from context) (p. 3)
$\operatorname{diag}(\cdot)$	diagonal matrix with specified entries (p. 4)
$A^0 = I$	convention for zeroth power of a matrix (p. 4)
A^{T}	transpose of A (p. 5)
$A^{-\mathsf{T}}$	inverse of A^{T} (p. 5)
$\overline{A}$	conjugate of A (p. 5)
A^*	conjugate transpose (adjoint) of A (p. 5)
A^{-*}	inverse of A^* (p. 5)
$\operatorname{tr} A$	trace of A (p. 6)
$\det A$	determinant of A (p. 8)
$\operatorname{adj} A$	adjugate of A (p. 9)
$\operatorname{sgn} \sigma$	sign of a permutation σ (p. 10)
$\deg p$	degree of a polynomial p (p. 12)
$\mathcal{P}_n$	set of complex polynomials of degree at most n (p. 21)

$\mathcal{P}_n(\mathbb{R})$	set of real polynomials of degree at most n (p. 21)
$\mathcal{P}$	set of all complex polynomials (p. 22)
$\mathcal{P}(\mathbb{R})$	set of all real polynomials (p. 22)
$C_{\mathbb{F}}[a, b]$	set of continuous $\mathbb{F}$-valued functions on $[a, b]$, $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ (p. 22)
$C[a, b]$	set of continuous $\mathbb{C}$-valued functions on $[a, b]$ (p. 22)
$\operatorname{null} A$	null space of a matrix A (p. 23)
$\operatorname{col} A$	column space of a matrix A (p. 23)
$\mathcal{P}_{\text{even}}$	set of even complex polynomials (p. 23)
$\mathcal{P}_{\text{odd}}$	set of odd complex polynomials (p. 23)
$A\mathcal{U}$	A acting on a subspace $\mathcal{U}$ (p. 23)
$\operatorname{span} \mathscr{S}$	span of a subset $\mathscr{S}$ of a vector space (p. 24)
$\mathbf{e}$	all-ones vector (p. 26)
$\mathcal{U} \cap \mathcal{W}$	intersection of subspaces $\mathcal{U}$ and $\mathcal{W}$ (p. 26)
$\mathcal{U} + \mathcal{W}$	sum of subspaces $\mathcal{U}$ and $\mathcal{W}$ (p. 27)
$\mathcal{U} \oplus \mathcal{W}$	direct sum of subspaces $\mathcal{U}$ and $\mathcal{W}$ (p. 27)
$\mathbf{v}_1, \mathbf{v}_2, \ldots, \widehat{\mathbf{v}_j}, \ldots, \mathbf{v}_r$	list of vectors with $\mathbf{v}_j$ omitted (p. 30)
$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$	standard basis of $\mathbb{F}^n$ (p. 35)
E_{ij}	matrix with (i, j) entry 1 and all others 0 (p. 35)
$\dim \mathcal{V}$	dimension of $\mathcal{V}$ (p. 35)
$[\mathbf{v}]_\beta$	coordinate vector of $\mathbf{v}$ with respect to a basis β (p. 40)
$\mathfrak{L}(\mathcal{V}, \mathcal{W})$	set of linear transformations from $\mathcal{V}$ to $\mathcal{W}$ (p. 41)
$\mathfrak{L}(\mathcal{V})$	set of linear transformations from $\mathcal{V}$ to itself (p. 41)
$\ker T$	kernel of T (p. 42)
$\operatorname{ran} T$	range of T (p. 42)
I	identity linear transformation (p. 44)
$\operatorname{row} A$	row space of a matrix A (p. 59)
$\operatorname{rank} A$	rank of a matrix A (p. 60)
$\star$	unspecified matrix entry (p. 65)
$A \oplus B$	direct sum of matrices A and B (p. 66)
$[A, B]$	commutator of A and B (p. 71)
$A \otimes B$	Kronecker product of matrices A and B (p. 74)
$\operatorname{vec} A$	vec of A (p. 75)
$\langle \cdot, \cdot \rangle$	inner product (p. 87)
$\perp$	orthogonal (p. 90)
$\lVert \cdot \rVert$	norm (p. 90)
$\lVert \cdot \rVert_2$	Euclidean norm (p. 91)
$\lVert \cdot \rVert_1$	ℓ^1 norm (absolute sum norm) (p. 97)
$\lVert \cdot \rVert_\infty$	ℓ^∞ norm (max norm) (p. 97)
${}_\gamma[T]_\beta$	matrix representation of $T \in \mathfrak{L}(\mathcal{V}, \mathcal{W})$ with respect to bases β and γ (p. 110)
F_n	$n \times n$ Fourier matrix (p. 129)
$\mathscr{U}^\perp$	orthogonal complement of a set $\mathscr{U}$ (p. 149)
$P_{\mathcal{U}}$	orthogonal projection onto $\mathcal{U}$ (p. 155)

$d(\mathbf{v}, \mathcal{U})$	distance from $\mathbf{v}$ to $\mathcal{U}$ (p. 160)
$G(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$	Gram matrix (p. 164)
$g(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$	Gram determinant (p. 164)
$\operatorname{spec} A$	spectrum of A (p. 183)
$\mathcal{E}_\lambda(A)$	eigenspace of A for eigenvalue λ (p. 186)
$p_A(\cdot)$	characteristic polynomial of A (p. 201)
$\mathcal{F}'$	commutant of a set of matrices $\mathcal{F}$ (p. 213)
e^A	matrix exponential (p. 212)
$m_A(\cdot)$	minimal polynomial of A (p. 229)
C_p	companion matrix of the polynomial p (p. 230)
$J_k(\lambda)$	$k \times k$ Jordan block with eigenvalue λ (p. 244)
J_k	$k \times k$ nilpotent Jordan block (p. 245)
$w_1, w_1, \ldots, w_q$	Weyr characteristic of a matrix (p. 252)
$\rho(A)$	spectral radius of A (p. 260)
$p(n)$	number of partitions of n (p. 271)
$\Delta(A)$	defect from normality of A (p. 285)
$A \circ B$	Hadamard product of A and B (p. 319)
$\lvert A\rvert$	modulus of A (p. 336)
$\sigma_{\max}(A)$	maximum singular value (p. 348)
$\sigma_{\min}(A)$	minimum singular value (p. 350)
$\sigma_1(A), \sigma_2(A), \ldots$	singular values of A (p. 350)
$A^\dagger$	pseudoinverse of A (p. 356)
$\kappa_2(A)$	spectral condition number of A (p. 359)
$\operatorname{Re} z$	real part of the complex number z (p. 398)
$\operatorname{Im} z$	imaginary part of the complex number z (p. 398)
$\lvert z\rvert$	modulus of the complex number z (p. 401)
$\arg z$	argument of the complex number z (p. 401)