0 Preliminaries

In this chapter, we review some concepts from elementary linear algebra and discuss mathematical induction. We document some facts about complex polynomials (including the Fundamental Theorem of Algebra, the division algorithm, and Lagrange interpolation) and introduce polynomial functions of a matrix.

0.1 Functions and Sets

Let $\mathscr{X}$ and $\mathscr{Y}$ be sets. The notation $f : \mathscr{X} \to \mathscr{Y}$ indicates that f is a *function* whose *domain* is $\mathscr{X}$ and *codomain* is $\mathscr{Y}$. That is, f assigns a definite value $f(x) \in \mathscr{Y}$ to each $x \in \mathscr{X}$. A function may assign the same value to two different elements in its domain, that is, $x_1 \neq x_2$ and $f(x_1) = f(x_2)$ is possible. But $x_1 = x_2$ and $f(x_1) \neq f(x_2)$ is not possible.

The *range* of $f : \mathscr{X} \to \mathscr{Y}$ is

$$\operatorname{ran} f = \{f(x) : x \in \mathscr{X}\} = \{y \in \mathscr{Y} : y = f(x) \text{ for some } x \in \mathscr{X}\},$$

which is a subset of $\mathscr{Y}$. A function $f : \mathscr{X} \to \mathscr{Y}$ is *onto* if $\operatorname{ran} f = \mathscr{Y}$, that is, if the range and codomain of f are equal. A function $f : \mathscr{X} \to \mathscr{Y}$ is *one to one* if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. Equivalently, f is one to one if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$; see Figure 0.1.

We say that elements $x_1, x_2, \ldots, x_k$ of a set are *distinct* if $x_i \neq x_j$ whenever $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$.

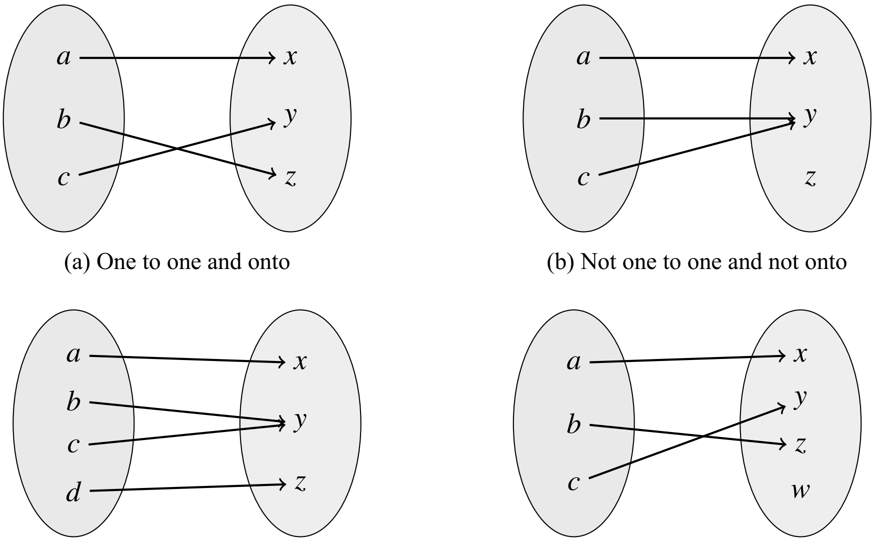
0.2 Scalars

We denote the real numbers by $\mathbb{R}$ and the complex numbers by $\mathbb{C}$. Real or complex numbers are called *scalars*. The only scalars that we consider are complex numbers, which we sometimes restrict to being real. See Appendix A for a discussion of complex numbers.

0.3 Matrices

An $m \times n$ *matrix* is a rectangular array

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{0.3.1}$$



(a) One to one and onto

(b) Not one to one and not onto

(c) Onto and not one to one

(d) One to one and not onto

Figure 0.1 Properties of functions: one to one and onto.

of real or complex numbers. The (i,j) *entry* of A is a_{ij}. Two matrices are *equal* if they have the same size (the same number of rows and columns) and if their corresponding entries are equal. An $n \times n$ matrix is a *square matrix*. The set of all $m \times n$ matrices with complex entries is denoted by $\mathsf{M}_{m\times n}(\mathbb{C})$, or by $\mathsf{M}_n(\mathbb{C})$ if $m = n$. For convenience, we write $\mathsf{M}_n(\mathbb{C}) = \mathsf{M}_n$ and $\mathsf{M}_{m\times n}(\mathbb{C}) = \mathsf{M}_{m\times n}$. The set of $m \times n$ matrices with real entries is denoted by $\mathsf{M}_{m\times n}(\mathbb{R})$, or by $\mathsf{M}_n(\mathbb{R})$ if $m = n$. In this book, we consider only matrices with real or complex entries.

Rows and Columns For each $i = 1, 2, \ldots, m$, the ith *row* of the matrix A in (0.3.1) is the $1 \times n$ matrix

$$[a_{i1}\ a_{i2}\ \ldots\ a_{in}].$$

For each $j = 1, 2, \ldots, n$, the jth *column* of A is the $m \times 1$ matrix

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

It is often convenient to write the matrix (0.3.1) as a $1 \times n$ array of columns

$$A = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n].$$

Addition and Scalar Multiplication If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then $A + B$ is the $m \times n$ matrix whose (i,j) entry is $a_{ij} + b_{ij}$. If $A \in \mathsf{M}_{m\times n}$ and c is a scalar, then $cA = [ca_{ij}]$ is the $m \times n$ matrix obtained by multiplying each entry of A by c. A *zero matrix* is an $m \times n$ matrix whose entries are all zero. Such a matrix is denoted by 0, although subscripts can be attached to indicate its size. Let $A, B \in \mathsf{M}_{m\times n}$ and let c, d be scalars.

(a) $A + B = B + A$.

(b) $A + (B + C) = (A + B) + C$.

(c) $A + 0 = A = 0 + A$.

(d) $c(A + B) = cA + cB$.

(e) $c(dA) = (cd)A = d(cA)$.

(f) $(c + d)A = cA + dA$.

Multiplication If $A = [a_{ij}] \in \mathsf{M}_{m\times r}$ and $B = [b_{ij}] \in \mathsf{M}_{r\times n}$, then the (i, j) entry of the product $AB = [c_{ij}] \in \mathsf{M}_{m\times n}$ is

$$c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}. \tag{0.3.2}$$

This sum involves entries in the ith row of A and the jth column of B. The number of columns of A must be equal to the number of rows of B. If we write $B = [\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_n]$ as a $1 \times n$ array of its columns, then (0.3.2) says that

$$AB = [A\mathbf{b}_1\ A\mathbf{b}_2\ \ldots\ A\mathbf{b}_n].$$

See Chapter 3 for other interpretations of matrix multiplication.

We say that $A, B \in \mathsf{M}_n$ *commute* if $AB = BA$. Some pairs of matrices in M_n do not commute. Moreover, $AB = AC$ does not imply that $B = C$. Let A, B, and C be matrices of appropriate sizes and let c be a scalar.

(a) $A(BC) = (AB)C$.

(b) $A(B + C) = AB + AC$.

(c) $(A + B)C = AC + BC$.

(d) $(cA)B = c(AB) = A(cB)$.

Identity Matrices The matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathsf{M}_n$$

is the $n \times n$ *identity matrix*. That is, $I_n = [\delta_{ij}]$, in which

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

is the *Kronecker delta*. If the size is clear from context, we write I in place of I_n. For every $A \in \mathsf{M}_{m\times n}$,

$$AI_n = A = I_m A.$$

Triangular Matrices Let $A = [a_{ij}] \in \mathsf{M}_n$. We say that A is *upper triangular* if $a_{ij} = 0$ whenever $i > j$; *lower triangular* if $a_{ij} = 0$ whenever $i < j$; *strictly upper triangular* if $a_{ij} = 0$ whenever $i \geq j$; and *strictly lower triangular* if $a_{ij} = 0$ whenever $i \leq j$. We say that A is *triangular* if it is either upper triangular or lower triangular.

Diagonal Matrices We say that $A = [a_{ij}] \in \mathsf{M}_n$ is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$. That is, any nonzero entry of A must lie on the *main diagonal* of A, which consists of the *diagonal entries* $a_{11}, a_{22}, \ldots, a_{nn}$; the entries a_{ij} with $i \neq j$ are the *off-diagonal entries* of A. The notation $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is used to denote the $n \times n$ diagonal matrix whose diagonal entries are $\lambda_1, \lambda_2, \ldots, \lambda_n$, in that order. A *scalar matrix* is a diagonal matrix of the form $\operatorname{diag}(c, c, \ldots, c) = cI$ for some scalar c. Any two diagonal matrices of the same size commute.

Superdiagonals and Subdiagonals The (first) *superdiagonal* of $A = [a_{ij}] \in \mathsf{M}_n$ contains the entries $a_{12}, a_{23}, \ldots, a_{n-1,n}$. The kth superdiagonal contains the entries $a_{1,k+1}, a_{2,k+2}, \ldots, a_{n-k,n}$. The kth *subdiagonal* contains the entries $a_{k+1,1}, a_{k+2,2}, \ldots, a_{n,n-k}$.

Tridiagonal and Bidiagonal Matrices A matrix $A = [a_{ij}]$ is *tridiagonal* if $a_{ij} = 0$ whenever $|i - j| \geq 2$. A tridiagonal matrix is *bidiagonal* if either its subdiagonal or its superdiagonal contains only zero entries.

Submatrices A *submatrix* of $A \in \mathsf{M}_{m \times n}$ is a matrix whose entries lie in the intersections of specified rows and columns of A. A $k \times k$ *principal submatrix* of A is a submatrix whose entries lie in the intersections of rows $i_1, i_2, \ldots, i_k$ and columns $i_1, i_2, \ldots, i_k$ of A, for some indices $i_1 < i_2 < \cdots < i_k$. A $k \times k$ *leading principal submatrix* of A is a submatrix whose entries lie in the intersections of rows $1, 2, \ldots, k$ and columns $1, 2, \ldots, k$. A $k \times k$ *trailing principal submatrix* of A is a submatrix whose entries lie in the intersections of rows $n-k+1, n-k+2, \ldots, n$ and columns $n-k+1, n-k+2, \ldots, n$.

Inverses We say that $A \in \mathsf{M}_n$ is *invertible* if there exists a $B \in \mathsf{M}_n$ such that

$$AB = I_n = BA. \tag{0.3.3}$$

Such a matrix B is an *inverse* of A. If A has no inverse, then A is *noninvertible*. Either of the equalities in (0.3.3) implies the other. That is, if $A, B \in \mathsf{M}_n$, then $AB = I$ if and only if $BA = I$; see Theorem 2.2.19 and Example 3.1.8.

Not every square matrix has an inverse. However, a matrix has at most one inverse. As a consequence, if A is invertible, we speak of *the* inverse of A, rather than *an* inverse of A. If A is invertible, then the inverse of A is denoted by A^{-1}. It satisfies

$$AA^{-1} = I = A^{-1}A.$$

If $ad - bc \neq 0$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{0.3.4}$$

For $A \in \mathsf{M}_n$, define

$$A^0 = I \quad \text{and} \quad A^k = \underbrace{AA\cdots A}_{k \text{ times}}.$$

If A is invertible, we define $A^{-k} = (A^{-1})^k$ for $k = 1, 2, \ldots$. Let A and B be matrices of appropriate sizes, let j, k be integers, and let c be a scalar.

(a) $A^j A^k = A^{j+k} = A^k A^j$.

(b) $(A^{-1})^{-1} = A$.

(c) $(A^j)^{-1} = A^{-j}$.

(d) If $c \neq 0$, then $(cA)^{-1} = c^{-1}A^{-1}$.

(e) $(AB)^{-1} = B^{-1}A^{-1}$.

Transpose The *transpose* of $A = [a_{ij}] \in \mathsf{M}_{m\times n}$ is the matrix $A^\mathsf{T} \in \mathsf{M}_{n\times m}$ whose (i, j) entry is a_{ji}. Let A and B be matrices of appropriate sizes and let c be a scalar.

(a) $(A^\mathsf{T})^\mathsf{T} = A$.

(b) $(A \pm B)^\mathsf{T} = A^\mathsf{T} \pm B^\mathsf{T}$.

(c) $(cA)^\mathsf{T} = cA^\mathsf{T}$.

(d) $(AB)^\mathsf{T} = B^\mathsf{T} A^\mathsf{T}$.

(e) If A is invertible, then $(A^\mathsf{T})^{-1} = (A^{-1})^\mathsf{T}$. We write $(A^{-1})^\mathsf{T} = A^{-\mathsf{T}}$.

Conjugate The *conjugate* of $A \in \mathsf{M}_{m\times n}$ is the matrix $\overline{A} \in \mathsf{M}_{m\times n}$ whose (i, j) entry is $\overline{a_{ij}}$, the complex conjugate of a_{ij}. Thus,

$$\overline{(\overline{A})} = A, \qquad \overline{A + B} = \overline{A} + \overline{B}, \qquad \text{and} \qquad \overline{AB} = \overline{A}\,\overline{B}.$$

If A has only real entries, then $A = \overline{A}$.

Conjugate Transpose The *conjugate transpose* of $A \in \mathsf{M}_{m\times n}$ is the matrix $A^* = \overline{A^\mathsf{T}} = (\overline{A})^\mathsf{T} \in \mathsf{M}_{n\times m}$ whose (i, j) entry is $\overline{a_{ji}}$. If A has only real entries, then $A^* = A^\mathsf{T}$. The conjugate transpose of a matrix is also known as its *adjoint*. Let A and B be matrices of appropriate sizes and let c be a scalar.

(a) $I_n^* = I_n$.

(b) $0_{m\times n}^* = 0_{n\times m}$.

(c) $(A^*)^* = A$.

(d) $(A \pm B)^* = A^* \pm B^*$.

(e) $(cA)^* = \overline{c}A^*$.

(f) $(AB)^* = B^*A^*$.

(g) If A is invertible, then $(A^*)^{-1} = (A^{-1})^*$. We write $(A^{-1})^* = A^{-*}$.

Special Types of Matrices Let $A \in \mathsf{M}_n$.

(a) If $A^* = A$, then A is *Hermitian*; if $A^* = -A$, then A is *skew Hermitian*.

(b) If $A^\mathsf{T} = A$, then A is *symmetric*; if $A^\mathsf{T} = -A$, then A is *skew symmetric*.

(c) If $A^*A = I$, then A is *unitary*; if A is real and $A^\mathsf{T}A = I$, then A is *real orthogonal*.

(d) If $A^*A = AA^*$, then A is *normal*.

(e) If $A^2 = I$, then A is an *involution*.

(f) If $A^2 = A$, then A is *idempotent*.

(g) If $A^k = 0$ for some positive integer k, then A is *nilpotent*.

Trace The *trace* of $A = [a_{ij}] \in \mathsf{M}_n$ is the sum of the diagonal entries of A:

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}.$$

Let A and B be matrices of appropriate sizes and let c be a scalar.

(a) $\operatorname{tr}(cA \pm B) = c \operatorname{tr} A \pm \operatorname{tr} B$.

(b) $\operatorname{tr} A^\mathsf{T} = \operatorname{tr} A$.

(c) $\operatorname{tr} \overline{A} = \overline{\operatorname{tr} A}$.

(d) $\operatorname{tr} A^* = \overline{\operatorname{tr} A}$.

If $A = [a_{ij}] \in \mathsf{M}_{m\times n}$ and $B = [b_{ij}] \in \mathsf{M}_{n\times m}$, let $AB = [c_{ij}] \in \mathsf{M}_m$ and $BA = [d_{ij}] \in \mathsf{M}_n$. Then

$$\operatorname{tr} AB = \sum_{i=1}^{m} c_{ii} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{m} b_{ji}a_{ij} = \sum_{j=1}^{n} d_{jj} = \operatorname{tr} BA. \tag{0.3.5}$$

Be careful: $\operatorname{tr} ABC$ need not equal $\operatorname{tr} CBA$ or $\operatorname{tr} ACB$. However, (0.3.5) ensures that

$$\operatorname{tr} ABC = \operatorname{tr} CAB = \operatorname{tr} BCA.$$

0.4 Systems of Linear Equations

An $m \times n$ *system of linear equations* (a *linear system*) is a list of linear equations of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array} \tag{0.4.1}$$

It involves m *linear equations* in the n *variables* (or *unknowns*) $x_1, x_2, \ldots, x_n$. The scalars a_{ij} are the *coefficients* of the system (0.4.1); the scalars b_i are the *constant terms*.

By a *solution* to (0.4.1) we mean a list of scalars $x_1, x_2, \ldots, x_n$ that satisfy the m equations in (0.4.1). A system of equations that has no solution is *inconsistent*. If a system has at least one solution, it is *consistent*. There are exactly three possibilities for a system of linear equations: it has no solution, exactly one solution, or infinitely many solutions.

Homogeneous Systems The system (0.4.1) is *homogeneous* if $b_1 = b_2 = \cdots = b_m = 0$. Every homogeneous system has the *trivial solution* $x_1 = x_2 = \cdots x_n = 0$. If there are other solutions, they are called *nontrivial solutions*. There are only two possibilities for a homogeneous system: it has infinitely many solutions, or it has only the trivial solution. A homogeneous linear system with more unknowns than equations has infinitely many solutions.

Matrix Representation of a Linear System The linear system (0.4.1) is often written as

$$A\mathbf{x} = \mathbf{b}, \tag{0.4.2}$$

in which

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \tag{0.4.3}$$

The *coefficient matrix* $A = [a_{ij}] \in \mathsf{M}_{m\times n}$ of the system has m rows and n columns if the corresponding system of equations (0.4.1) has m equations in n unknowns. The matrices $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ and $m \times 1$, respectively. Matrices such as $\mathbf{x}$ and $\mathbf{b}$ are *column vectors*. We sometimes denote $\mathsf{M}_{n\times 1}(\mathbb{C})$ by $\mathbb{C}^n$ and $\mathsf{M}_{n\times 1}(\mathbb{R})$ by $\mathbb{R}^n$. When we need to identify the entries of a column vector in a line of text, we often write $\mathbf{x} = [x_1\ x_2\ \ldots\ x_n]^\mathsf{T}$ instead of the tall vertical matrix in (0.4.3).

An $m \times n$ homogeneous linear system can be written in the form $A\mathbf{x} = \mathbf{0}_m$, in which $A \in \mathsf{M}_{m\times n}$ and $\mathbf{0}_m$ is the $m \times 1$ column vector whose entries are all zero. We say that $\mathbf{0}_m$ is a *zero vector* and write $\mathbf{0}$ if the size is clear from context. Since $A\mathbf{0}_n = \mathbf{0}_m$, a homogeneous system always has the trivial solution.

If $A \in \mathsf{M}_n$ is invertible, then (0.4.2) has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Reduced Row Echelon Form Three elementary operations can be used to solve a system (0.4.1) of linear equations:

(I) Multiply an equation by a nonzero constant.

(II) Interchange two equations.

(III) Add a multiple of one equation to another.

One can represent the system (0.4.1) as an *augmented matrix*

$$[A\ \mathbf{b}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \tag{0.4.4}$$

and perform *elementary row operations* on (0.4.4) that correspond to the three permissible algebraic operations on the system (0.4.1):

(I) Multiply a row by a nonzero constant.

(II) Interchange two rows.

(III) Add a multiple of one row to another.

Each of these operations is reversible.

The three types of elementary row operations can be used to *row reduce* the augmented matrix (0.4.4) to a simple form from which the solutions to (0.4.1) can be obtained by inspection. A matrix is in *reduced row echelon form* if it satisfies the following:

(a) Rows that consist entirely of zero entries are grouped together at the bottom of the matrix.

(b) If a row does not consist entirely of zero entries, then the first nonzero entry in that row is a one (a *leading one*).

(c) A leading one in a higher row must occur further to the left than a leading one in a lower row.

(d) Every column that contains a leading one must have zero entries everywhere else.

Each matrix has a unique reduced row echelon form.

The number of leading ones in the reduced row echelon form of a matrix is equal to its rank; see Definition 2.2.6. Other characterizations of the rank are discussed in Section 3.2. It is always the case that $\operatorname{rank} A = \operatorname{rank} A^\mathsf{T}$; see Theorem 3.2.1.

Elementary Matrices An $n \times n$ matrix is an *elementary matrix* if it can be obtained from I_n by performing a single elementary row operation. Every elementary matrix is invertible; the inverse is the elementary matrix that corresponds to reversing the original row operation. Multiplication of a matrix on the left by an elementary matrix performs an elementary row operation on that matrix. Here are some examples:

(I) Multiply a row by a nonzero constant:

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

(II) Interchange two rows:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}.$$

(III) Add a nonzero multiple of one row to another:

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} \\ a_{21} & a_{22} \end{bmatrix}.$$

Multiplication of a matrix on the right by an elementary matrix corresponds to performing column operations. An invertible matrix can be expressed as a product of elementary matrices.

0.5 Determinants

The determinant function $\det : \mathsf{M}_n(\mathbb{C}) \to \mathbb{C}$ is of great theoretical importance, but of limited numerical use. Computation of determinants of large matrices should be avoided in applications.

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Excerpt
More Information

Laplace Expansion We can compute the determinant of an $n \times n$ matrix as a certain sum of determinants of $(n-1) \times (n-1)$ matrices. Let $\det[a_{11}] = a_{11}$, let $n \geq 2$, let $A \in \mathsf{M}_n$, and let $A_{ij} \in \mathsf{M}_{n-1}$ denote the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j of A. Then for any $i, j \in \{1, 2, \ldots, n\}$, we have

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det A_{ik} = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det A_{kj}. \tag{0.5.1}$$

The first sum is the *Laplace expansion by minors along row i* and the second is the *Laplace expansion by minors along column j*. The quantity $\det A_{ij}$ is the (i, j) *minor of* A; $(-1)^{i+j} \det A_{ij}$ is the (i, j) *cofactor* of A.

Using Laplace expansions, we compute

$$\begin{aligned}\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= a_{11} \det[a_{22}] - a_{12} \det[a_{21}] \\ &= a_{11}a_{22} - a_{12}a_{21}\end{aligned}$$

and

$$\begin{aligned}\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} \\ &\quad - a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21}.\end{aligned}$$

Following the same rule, the determinant of a 4×4 matrix can be written as a sum of four terms, each involving the determinant of a 3×3 matrix.

Determinants and the Inverse If $A \in \mathsf{M}_n$, the *adjugate* of A is the $n \times n$ matrix

$$\operatorname{adj} A = [(-1)^{i+j} \det A_{ji}],$$

which is the transpose of the matrix of cofactors of A. The matrices A and $\operatorname{adj} A$ satisfy

$$A \operatorname{adj} A = (\operatorname{adj} A)A = (\det A)I. \tag{0.5.2}$$

If A is invertible, then

$$A^{-1} = (\det A)^{-1} \operatorname{adj} A. \tag{0.5.3}$$

Properties of Determinants Let $A, B \in \mathsf{M}_n$ and let c be a scalar.

(a) $\det I = 1$.

(b) $\det A \neq 0$ if and only if A is invertible.

(c) $\det AB = (\det A)(\det B)$.

(d) $\det AB = \det BA$.

(e) $\det(cA) = c^n \det A$.

(f) $\det \overline{A} = \overline{\det A}$.

(g) $\det A^{\mathsf{T}} = \det A$.

(h) $\det A^* = \overline{\det A}$.

(i) If A is invertible, then $\det(A^{-1}) = (\det A)^{-1}$.

(j) If $A = [a_{ij}] \in \mathsf{M}_n$ is upper or lower triangular, then $\det A = a_{11}a_{22}\cdots a_{nn}$.

(k) $\det A \in \mathbb{R}$ if $A \in \mathsf{M}_n(\mathbb{R})$.

Be careful: $\det(A + B)$ need not equal $\det A + \det B$. Property (c) is the *product rule* for determinants.

Determinants and Row Reduction The determinant of an $n \times n$ matrix A can be computed with row reduction and the following properties:

(I) If A' is obtained by multiplying each entry of a row of A by a scalar c, then $\det A' = c \det A$.

(II) If A' is obtained by interchanging two different rows of A, then $\det A' = -\det A$.

(III) If A' is obtained from A by adding a scalar multiple of a row to a different row, then $\det A' = \det A$.

Because $\det A = \det A^{\mathsf{T}}$, column operations have analogous properties.

Permutations and Determinants A *permutation* of the list $1, 2, \ldots, n$ is a one-to-one function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. A permutation induces a reordering of $1, 2, \ldots, n$. For example, $\sigma(1) = 2$, $\sigma(2) = 1$, and $\sigma(3) = 3$ defines a permutation of $1, 2, 3$. There are $n!$ distinct permutations of the list $1, 2, \ldots, n$.

A permutation $\tau : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ that interchanges precisely two elements of $1, 2, \ldots, n$ and leaves all others fixed is a *transposition*. Each permutation of $1, 2, \ldots, n$ can be written as a composition of transpositions in many different ways. However, the parity (even or odd) of the number of transpositions involved depends only upon the permutation. We say that a permutation σ is *even* or *odd* depending upon whether an even or odd number of transpositions is required to represent σ. The *sign* of σ is

$$\operatorname{sgn}\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

The determinant of $A = [a_{ij}] \in \mathsf{M}_n$ can be written

$$\det A = \sum_{\sigma} \left(\operatorname{sgn}\sigma \prod_{i=1}^{n} a_{i\sigma(i)} \right),$$

in which the sum is over all $n!$ permutations of $1, 2, \ldots, n$.

Determinants, Area, and Volume If $A = [\mathbf{a}_1\ \mathbf{a}_2] \in \mathsf{M}_2(\mathbb{R})$, then $|\det A|$ is the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ (its vertices are at $\mathbf{0}$, $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_1 + \mathbf{a}_2$). If $B = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3] \in \mathsf{M}_3(\mathbb{R})$, then $|\det B|$ is the volume of the parallelopiped determined by $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$.