

INDEX

- Abelian groups, 4, 6–8, 37, 88
- Abscissa, 113, 410. *See also* *x-axis*
- Absolute hypersphere, 15–17, 49, 51, 53, 65, 156, 362, 373. *See also* *Absolute sphere*
- Absolute involution, 49, 230
- Absolute k -planes (\mathbb{E}^k), of complete hyperbolic n -space, 17
- Absolute points, 15–17, 24, 49, 51, 56, 84–85, 161, 224, 226
- Absolute polarities, xii, 17, 27, 45–47, 50, 134
- Absolute sphere, Eisenstein integer lattice on, 343
- Absolute values, of quaternions, 212
- Acrorectangular group ([1, 2]), 231, 233
- Acrorhombic group ([1, 2]⁺), 231, 236
- Acrotrigonal group, full, 241–242
- Affine geometry, xi, 27, 86, 395
- Affine groups, 181–182, 408–409
- Affine spaces (A^n), 14, 87, 132–134. *See also* *Affine line(s); Affine n -space (A^n , FA^n); Affine plane (A^2); Complex affine n -space ($\mathbb{C}A^n$)*
- Affine transformations, 149, 168–169, 171–174
- Affinities, 87, 92–93, 95, 113, 116–117, 132–137, 168, 172–174, 181–182, 280–283, 285–288
- Algebra(s), xi, 1–2, 9, 333, 349, 378. *See also* *Clifford algebras; Linear algebra*
- Algebraic integers, 332
- Alternating groups (A_n), 5, 241, 280–281
- Alternative division rings, 37–38, 378
- Anasymmetric groups, 249–250, 295, 410
- Annihilator(s), 9, 326
- Antiholomorphisms, of Möbius n -antisphere, 201–202
- Anti-involution(s), 25, 157–158
- Antilinear forms, 169
- Antilinear geometries, 157–162, 166
- Antilinear transformations, 168, 169–170, 174, 341, 408
- Antipodal points, xi, 3, 14, 18–19, 60–65, 71–77, 163, 165
- Antipolarities, 25, 157–161, 179
- Antiprojectivities, 183, 197–198, 215–217
- Antiquadratic forms, 162, 204
- Antiscalar transformations, 170
- Antispheres, 157, 163–166. *See also* *Elliptic n -antisphere; Hyperbolic n -antisphere; Inversive (n -1)-antisphere; Möbius n -antisphere (Iu^n)*
- Anti-unitary transformations, 205
- Apeirogonal group(s) ([∞]), 228–229, 263, 266–269

- Apeirohedral groups, 271–278, 309, 373
- Apeirohedron, hyperbolic, 238
- Aperiodic isometries, 156
- Aperiodic operations, 263
- Apex (apices), 231, 234, 237
- Areplectic geometry (Ap^n), real transformation groups in, 395
- Areplectic groups (Ap_n), translation group *vs.*, 133
- Argand diagram (Eu^1), xii, 25, 162, 195.
See also Complex field (\mathbb{C})
- Automorphism groups, 359–361, 371, 375
- Axial dions, 234
- Axial symmetries, 244
- Axiom of completion, 37
- Ball, as model of hyperbolic n -space, 50
- Barycenters, 87, 91, 95, 174–175
- Basepoints (Δ_i), 91, 113, 137, 143
- Basic sets, of octonionic integers, 379–383
- Beltrami–Klein model, 50–51, 53, 70–71
- Bicuspal simplexes, as fundamental regions, 327–328
- Bilateral group ([]), 227, 231, 233, 245, 254, 410
- Bilinear forms, 8–9, 44–51, 58, 60, 63, 66, 68, 132, 135, 138, 185, 191–192, 322, 408
- Bipartite groups, 261
- Bisymplectic group (Sp'_n), 135–136
- Bivalent matrices, 170–171, 175, 212
- Boundaries, of hyperbolic unitary n -space, 211
- Bounded fundamental regions, of hyperbolic Coxeter groups, 308
- Branches, in Coxeter–Dynkin diagrams, 298
- Cartan simplex(es), 289–290, 391
- Cartesian coordinates, 47–48, 64–67, 69–70, 143, 160, 185–187, 194
- Cellular polytopes, 365, 381–382, 384
- Cellulations, 225, 305–306, 308, 338–339, 343–344, 362, 376–377, 410
- Cellules, of an n -honeycomb, 225
- Central group, 233–234, 245, 256–257
- Centrally symmetric hexagons, torohedral groups and, 278
- Central n -sphere, 18
- Ceva's Theorem, 115–117
- Chain(s), 10, 25, 36–37, 157–158, 160–163, 183, 196–197, 200–202, 215–216, 410
- Change of coordinates, in affine spaces, 93
- Characteristic of a ring, 38
- Circular geometries, i, xi–xii, 27, 56–86
- Circularity group (s), 183, 395, 390
- Circular metric spaces, 27, 57
- Classical groups, xiii
- Classical Lie algebras, dimensions of, 391
- Clifford algebras, 218–220, 355
- Co-automorphism(s), 44, 109, 212
- Co-Euclidean geometry, 17
- Collinearity, 1–3, 68, 132–133
- Collineation groups, 87–112, 168–182, 390, 395
- Commutators, 4–5, 104–105
- Compact groups, defined, 304
- Complex conformal n -space, 26
- Complex conjugation, 13, 25, 26, 204–206
- Complex geometries, 10, 157
- Complex groups, octonionic modular loops and, 387–388
- Complex numbers (\mathbb{C}), xii, 6, 12, 38, 194, 196, 333–334, 342, 349, 378–379, 382–383, 390, 407
- Complex projective chains, real inversive circles *vs.*, 200
- Compound Eisenstein integers, 383–386
- Compound polygons, hexagram as, 255
- Concatenation group, of complex projective line, 183, 198, 2–3
- Concyclic points, 2, 163, 165–166, 187–188

- Conformal geometry, 60
- Conformal models, 57, 65–74, 195, 201, 215–216. *See also* Conformal ball model; Conformal disk model; Conformal double half-space model
- Conjugate(s), 378, 408
- Connectivity, 71
- Continuous groups, i, xiii, 12, 223, 295, 406, 408–410
- Convex polytopes, finitary Coxeter groups and, 304–308
- Coordinatization, xi, xii, 1, 9–11, 38, 42–43
- Coxeter groups, i, xiii, 223, 227, 239–242, 246–263, 271–278, 289–330, 332, 334–336, 339, 341–342, 347–349, 356–357, 361–364, 371–377, 381, 388, 391–393, 396
- Cross ratios, xi, 41–43, 46, 49–50, 52, 59, 87, 101–102, 105–106, 109, 157–158, 163–165, 175–178, 196–197, 216, 280, 407
- Crystallographic Coxeter groups, 296–297
- Crystallographic groups, 289–290, 391
- Curvature, 13–14, 71
- Cusps, 300
- Cyclosymmetric group(s) ($S_{[n+1]}$), 292, 295
- Cyclotomic group ($C_m(\mathbb{C})$), 204
- Degenerate hyperspheres, 58, 60, 62
- Degenerate paratax, in affine plane, 117
- Demibisymmetric group(s) (\mathbf{D}_n), 250–251, 410
- Demiorthocombs ($h\delta_{n+1}$), 293–294
- Demiorthotope(s) ($h\gamma_n$), 251
- Desargues configuration, 32–33, 100–102
- Desarguesian planes, 32–35, 38, 42
- Desargues's Theorem, 32–34, 37–39, 42–43, 46
- Determinant(s), 90, 106, 132, 139, 204, 213, 219–221, 247, 391
- Diagonal matrices, 89, 408
- Diagonal points, of a complete quadrangle, 29, 39
- Diagonals, of quadrilaterals, 237
- Differentiable n -manifolds, 135
- Dihedral groups ($[p]$, D_p , $I_2[p]$), 5, 228–229, 255, 258–259, 299
- Dilatation(s), 19, 94–96, 125, 148–150, 173–174, 189, 208
- Dilative groups, 174, 409
- Dion(s) ($\{\}$), 224–227, 232, 234, 237, 251, 299, 391, 410. *See also* Line segment(s)
- Direct elliptic isometry groups (P^+O_{n+1}), 395
- Direct Euclidean (isometry) group(s) (E_n^+), 124, 145, 180, 395, 409
- Direct hyperbolic isometry groups ($P^+O_{n,1}$), 395
- Direct isometries, 19–21, 128, 145, 147, 152–156, 190, 199, 214, 217–218, 227, 254, 280–281, 286–288, 302, 357–358
- Direct linear groups (G^+L_n), 90, 111–112, 132, 409
- Direct orthopetic group(s) (\tilde{O}_n^+), 149, 181, 409
- Direct pseudospherical isometry groups ($O_{n,1}^+$), 395
- Direct spherical isometry groups (O_{n+1}^+), 395
- Dirichlet L -functions, 394, 405
- Discrete groups, i, xiv, 223, 248, 297–298, 299, 303, 376, 406, 408–410
- Discriminant(s), 51–52, 58–59, 63, 117, 161
- Disphenoid, 237–238, 251
- Distance functions, in unitary spaces, 13
- Diverging hyperplanes, 51–52, 69, 154, 161, 407
- Division algebras, 9, 349, 378
- Division algorithm, 10–11
- Division rings, 12, 37–39, 109, 203, 211, 220, 349, 355, 378, 383–384, 407–408
- Dodecahedron (dodecahedra; {5, 3}), 240, 248–249, 308–310

- Domains, 6, 11, 407
- Double groups, 255, 258, 260–262
- Double half-space model, 65
- Double Hurwitzian system ($\mathbb{O}\text{ch}$), 382–383
- Double prism(atic) honeycombs, 352, 373, 383
- Double projective general linear group (DPGL_n), 107–108
- Double rotary reflections, 22
- Double rotation(s), 22, 214
- Doubly asymptotic triangles, as fundamental regions, 300–301
- Dual bilinear forms, 46–47, 51
- Duality, xii, 2, 28, 32, 45, 88, 106–107, 309–310, 406
- Dualization, 9, 28
- Dual regular polytopes, Schläfli symbols for, 226
- Duals, 234, 237
- Dual vector spaces, xii, 9, 40–41, 169–170
- Duplex coordinates, 171–172, 206–207
- Duplex groups, 295
- Duplex matrices, 168, 170–171, 203, 212–214
- Edges, of polytopes, 224–225
- Eigenvectors, 103, 139–143, 145–147, 153–154
- Eisenstein extended modular group ($\text{P}\tilde{\text{S}}\text{L}_2(\mathbb{E})$), 343; Eisenstein (complex) integer (ring) (\mathbb{E}), 342–343, 352, 378, 383–386, 407; Eisenstein integer matrices ($\text{SL}_2(\mathbb{E})$), 342, 346; Eisenstein modular group(s) ($\text{PSL}_2(\mathbb{E})$), 342–348, 376–377; Eisenstein octaves ($\mathbb{O}\text{ce}$), 384–387; Eisenstein special linear group(s) ($\text{SL}_2(\mathbb{E})$), 342, 346; Eisenstein units ($\tilde{\text{S}}\text{L}(\mathbb{E})$), 342, 383–385
- Elation(s), 43, 105, 125
- Elative homographies, 193
- Ellipses, 45, 116–117, 119–120, 131
- Elliptic geometry ($e\text{P}^n$), i, xii, 15, 17, 45–50, 55–56, 65–67, 86, 128, 395
- Elliptic polarity (polarities), 18, 25, 47, 49, 109
- Elliptic rotations, 119–121, 130–131
- Elliptic unitary line (Pu^1), 25–26, 162
- Epimorphisms, defined, 7
- Epsilon (\mathbb{E}) reference points, in projective spaces, 96–97. *See also* Unit points
- Equator, 72, 75. *See also* Equatorial entries
- Equatorial circle, 3, 63
- Equatorial hypersphere, 62, 64–65
- Equiaffinity (equiaffinities), 93, 95, 113, 116–117, 123, 125, 127, 129–131, 133, 137
- Equiangular polygons, 226
- Equiareal collineations, 113–137
- Equidistant curve(s), 14–15, 47, 53, 83–84
- Equidistant k -surface, 15. *See also* Equidistant surface
- Equilateral polygons, 226
- Equiplexities, 113, 133, 137
- Equipositive pseudospherical isometry groups ($\text{O}_{n+,1+}$), 395
- Equiprojective collineations, 104–105
- Equivalence relation, similarity as, 106
- Erlanger Programm* (Klein), i, xii–xiii, 18
- Euclidean Coxeter groups, xiii, 223; Euclidean symmetry groups, 263–298; Euclidean tessellations, 238–239; Euclidean transformations, 143–150, 203; Euclidean unitary group, 207–208; Euclidean unitary n -space (Eu^n), 24–25, 157, 159–160, 206–208, 407
- Euclid's Fifth Postulate, 84. *See also* Parallel postulate
- Euclcloptic groups ($\check{\text{E}}_n$), 19, 148, 189, 409
- Exceptional Euclidean groups, 318–321
- Exceptional groups, 294, 297, 410
- Exceptional hypersphere, of parabolic n -sphere, 184

- Extended affine n -space (\bar{FA}^n), 47–48
- Extended Euclidean spaces, 15–16
- Extended Euclidean unitary n -space (\bar{EU}^n), 159–160
- Extended group(s), 244–245, 249–250, 258, 264–266, 375
- Extended hyperbolic spaces (\bar{H}^n), 15–18
- Extensions, 223, 254–262, 268–269, 390, 407
- Faces, 225, 237
- Factor groups, defined, 4
- Factorization, 11
- Fano configuration, 29–30, 39, 284
- Fields, 6, 8–9, 39–40, 42, 44–45, 87–91, 98, 101, 109, 407
- Finitary hyperbolic groups, 299, 308
- Finite crystallographic reflection groups, 295
- Finite fields, 407
- Finite irreducible crystallographic groups, Lie algebras *vs.*, 391
- Finite isometry groups, in 3-space, 231, 244–245
- Forms, 8–9, 44–45, 408
- Fractional transformations, xiv, 408
- Frames of reference (Δ), 96–99, 102–103, 106–107, 109–110, 114–116
- Free groups, 229
- Frieze patterns, 263–266
- Full apeirogonal group(s), 228–229, 263, 290
- Full gyropentagonal group, 242
- Full orthorhombic group ([2, 2]), 232
- Full orthotetragonal group ([2, 4]), 242
- Full pseudogonal group ($[\pi i/\lambda], \tilde{\mathbf{A}}[\lambda]$), 229–230, 299
- Full symmetry groups, 5, 276
- Full trigonohedral group (**p3m1**), 271
- Fundamental tetrahedra, 310–311
- Fundamental Theorem of Arithmetic, 11
- Fundamental Theorem of Algebra, 25
- Fundamental Theorem of Projective Geometry, 44, 102
- Fusil(s), 234–235, 238, 251–252, 410
- Gamma function (γ), 392
- Gaussian integers (\mathbb{G}), 337–339, 352, 378, 380–381, 384–385. *See also* Gaussian complex integers (\mathbb{G})
- General dilative group(s) (GD_n), 95, 110–112, 148, 174, 409
- General linear groups (GL_n , $GL_n(\mathbb{R})$), 11, 90, 94, 104, 110–112, 170, 212–213, 331, 390, 406, 408–409
- General semiscalar group ($GZ(\mathbb{C})$), 170
- General translinear group (TGL_n), 96
- Geodesic thread, 163
- Geometric transformations, xiii, 11–12, 18
- Glide reflection(s), 22, 263, 265
- Golden ratio, 230
- Gosset groups, 410
- Gosset polytopes, 252
- Gram matrices, 247, 254, 295, 303
- Graph amalgamation product, 360
- Graphs, Coxeter diagrams as, 246–247, 249–252
- Gravesian modular loop ($PSM_2[\mathcal{Ocg}]$), 387
- Gravesian octaves (\mathcal{Ocg}), 379–381, 385–386, 387
- Gravesian special Moufang loop ($SM_2[\mathcal{Ocg}]$), 387
- Great hyperchains, 163–164, 165, 204–206, 210
- Grid honeycomb(s), 294, 410
- Gyral pairs, 132–133
- Gyropetic groups (Gp_n), 132, 136, 215, 409
- Gyrorectangular group ([2+, 4]), 236, 241–242
- Gyro-tetragonohedral group (**p4g**), 272–273
- Gyro-trigonohedral group (**p3m**), 273
- Hamilton group (**2D₂**), 351–352
- Hamilton integers (\mathbb{Ham} ; $\mathbb{Ham}[ijk]$), 352–356, 360–361, 364–365, 367,

- 378–381, 385–386, 407. *See also* Hamilton quaternionic integers ($\mathbb{H}\text{am}$)
- Hamilton modular group ($\text{PSL}_2(\mathbb{H}\text{am})$), 364–368, 371, 375, 377
- Harmonic conjugates, 29, 49
- Harmonic homology (homologies), 44, 59, 61, 101, 105, 125, 163
- Harmonic sets, 27, 29–34, 42–43, 87, 105, 109, 157, 410
- Hasse diagrams, 224, 225
- Hecatonicosachoron. *See* Dodecacontachoron ($\{5.3.3\}$)
- Heisenberg group, 162
- Heron’s Formula, 81
- Hexagonal tessellation ($\{6, 3\}$), symmetry group of, 271–272, 277–278
- Hexagonohedral group ($\mathbf{p6}$), 272
- Holomorphisms, 163, 201–202
- Homogeneous coordinates, 39–42, 48–49, 57–58, 60, 62, 66, 68, 96–102, 113–117, 141–142, 150, 152–153, 158–160, 162–164, 184–186, 194, 196, 200, 204–206, 210, 285–288
- Homographies, 12, 156, 183, 192–193, 198–200, 202, 217–218, 220
- Homology (homologies), 43–44, 59, 105, 163–165
- Homomorphisms, 6–8, 90, 201, 214
- Homothetic transformation, 94
- Honeycombs, 223, 225–226, 292–295, 307–308, 313, 315–318, 345, 376–386, 406–407, 410. *See also* Cellulations; Lattice entries; n -honeycombs; Spherical cellulations; Tessellations; Uniform honeycombs; Uniform tilings
- Hopf fibration, 164
- Horizontal lines, in extended affine plane, 114
- Horocycles, 14, 24, 53–56, 63, 84–86, 185, 226, 229
- Horospherical tessellations, 338, 343–344
- Hurwitz group ($2A_4$), 351–352
- Hurwitz integers ($\mathbb{H}\text{ur}[ijk]$), 352–353, 360–361, 368–370, 378–379, 380–383, 407
- Hurwitz modular group ($\text{PSL}_2(\mathbb{H}\text{ur})$), 368–372, 377
- Hybrid integers ($\mathbb{H}\text{yb}; \mathbb{H}_{32}$), 352–353, 361–362, 372–375, 378, 383–386. *See also* Hybrid quaternionic integers ($\mathbb{H}\text{yb}$)
- Hybrid modular group ($\text{PSL}_2(\mathbb{H}\text{yb})$), 372–377
- Hyperbolas, 45, 116–117, 127–129, 131
- Hyperbolic circle (\mathring{S}^1), 128; Hyperbolic Coxeter groups, xiii–xiv, 223, 297, 299–329, 361–362, 388, 392; Hyperbolic geometry (H^n), i, xii, 15, 17, 50–56, 67–71, 86, 128, 322; Hyperbolic groups, 296–297, 294–301, 304–308, 311–313, 323–383, 391–393, 400–405; Hyperbolic isometry group(s), 152, 395; Hyperbolic k -space (H^k), 15; Hyperbolic line(s) (H^1), 69–70, 128, 226, 229–230, 299; Hyperbolic polyhedra, 311–313; Hyperbolic reflection groups, existence in higher-dimensional spaces, 308; Hyperbolic Schläfli function (iS), 393; Hyperbolic sectors, 127; Hyperbolic tessellations, 238–239, 300–301, 304, 332–333, 335–336; Hyperbolic 3-space (H^3), 54–55, 305, 309–313, 338–339, 343–345, 376–377; Hyperbolic triangle(s), 74, 81, 300–302, 304; Hyperbolic unitary n -space (Hu^n), 24, 25, 157, 161, 210, 407; Hyperbolic unitary space, 166
- Hypercompact Coxeter groups, 313, 339, 362–364, 373–374, 376–377
- Hypercompact groups, defined, 304
- Hypercompact subgroups, 358, 388, 367
- Hyperplane at infinity, 18, 45–47, 133–134, 164

- Hyperplanes, 8, 14–15, 17, 20–21, 23, 39–49, 52, 58, 60–63, 65–66, 68, 87, 91, 93, 103, 106–110, 142–147, 150, 158–162, 164–166, 171–172, 174–178, 184, 186, 206–207, 210, 216, 289–290, 322–323, 407, 408. *See also* Hyperplanes in hyperbolic n -space; Ideal hyperplanes; Ideal k -planes
- Hypersphere(s), 14–16, 20, 54–55, 58–65, 71, 200, 202, 216, 362
- Icosahedral group ($[3, 5]^+$), 239, 243–245, 248–249
- Icosahedron (icosahedra; $\{3, 5\}$), 240, 248–249
- Ideal k -planes, of complete hyperbolic n -space, 17
- Ideal lines, of complete hyperbolic n -space, 17
- Ideal points, 17, 51, 160–161, 230, 326
- Identity (1), 20–21, 88, 108–109, 111–112.
- Identity automorphism, 7
- i -faces, of an n -polytope, 224
- Image (Img), transformations and, 7, 90, 91
- Incidence, 27–46, 75, 102, 175–177, 410
- Incidence-preserving transformations, 87
- Index, of a subgroup, 4
- Infinite Coxeter groups, 223, 249–250, 290, 299
- Infinite cyclic group (C_∞), 229–230
- Infinite dihedral group (D_∞, \bar{A}_1), 229–230, 299
- Infinite fundamental regions, 289–290
- Infinitesimal quantities, 10
- Inner automorphism(s), 7, 212, 216
- Inner product(s), 50, 88, 138–139, 204
- Inner-product spaces, Euclidean metric and, 48
- Integral domain(s), 6, 11, 333, 337, 342, 345
- Integral octonions, 349, 378–387
- Integral quaternions, 349–355, 385–386
- Interior angles, 3, 76–78
- Intersecting hyperplanes, 51–52, 69, 161, 407
- Inverse, of an automorphism, 7
- Inverse dual linear transformations, 103
- Inverse dual transformations, of projective linear space, 176
- Inverse elements, 4, 6
- Inverse translinear transformations, 93
- Inversive-disk model, projective-disk model *vs.*, 70–71, 74
- Inversive geometry, i, xi, xii, 4, 16, 27, 56–61, 86, 395
- Inversive n -space (I^n), xii, 18, 23, 27, 395, 407
- Inversive unitary geometry, 157, 162, 199–202
- Inversive unitary n -space (Iu^n), 26, 183, 200–201, 407
- Involutions, 18, 34, 44, 98, 109
- Ionic subgroups, 239–242, 244, 257–261, 274, 339, 358, 366, 370–371
- Irreducible Coxeter groups, 247–249, 252, 263, 299
- Irreducible Euclidean groups, 290, 292–295
- Irreducible higher-dimensional finite symmetry groups, 260–261
- Irreducible quadratic polynomials, 25–26
- Isobaric figures, 95
- Isobaric transformation, 93
- Isometric models, for hyperbolic n -space, 64–65
- Isometry (isometries), i, 13–14, 18–24, 128, 138–143, 145, 150–156, 189–190, 199, 203, 205–208, 210, 211, 214, 218, 220, 223, 225–227, 285–289, 330, 377, 390
- Isometry groups, 18–20, 138–358, 390, 395
- Isoplectic group(s) ($\bar{S}Ap_n$), 134
- Isoscles, 86, 231–237
- Isotropic, 13–14, 133
- Join operation, 28, 251–252, 410, 6–7, 90–91, 104–105

- Kites, 237
- Kleinian octaves, 382, 386
- Koszul simplexes, 305, 308, 328, 391
- Kronecker delta (δ_{ij}), 91
- k -spheres (S^k), 14–15, 65, 67, 71
- Lannér simplexes, 305, 308, 328, 391
- Lateral rectangles, 232–233
- Lattice(s), 24, 331, 337–339, 343, 349–352, 356, 361–362, 365, 369, 373, 379, 381–386
- Lattice groups, torohedral, 278–289
- Lattice patterns, 266–271
- Lattice points, on a torus, 278–279
- Left double rotations, quaternions as, 214
- Length, 1–2, 48, 79–80, 85–86, 322, 328
- L -functions, 394, 405
- Lie algebras, 391
- Lie groups, xiii, 248, 296, 298
- Light cones, 64
- Limiting curve, 53
- Limiting surface, 54
- Linear associative algebras, 218
- Linear combinations, of vectors, 8, 88–89
- Linear complex, 134
- Linear dependence, 132–133
- Linear forms, 8, 9, 88, 168–169. *See also* Bilinear forms
- Linear fractional transformations, i, xiv, 11, 197–199, 216–217, 218, 220, 302–303, 330, 332, 336, 345, 348–349, 357, 362–363, 366, 368–369, 370, 372, 374, 376–378, 387, 390, 408
- Linear geometries, i, xi–xii, 27–56, 65–71, 181–182, 330–332, 406, 408–409
- Linear independence, 8, 88, 102
- Linearity, 89–90
- Linear metric spaces, 27, 45–46, 57
- Linear rotation matrices, 118–121, 129–130
- Linear semigroups, 89–90, 409
- Linear spaces, projective, 102
- Linear transformations, 8, 11, 87–92, 102–103, 168–171, 181–182, 408
- Line(s) at infinity, 2, 125, 114, 285
- Line segment(s), 74–75, 81, 115, 223–224, 303, 322, 410
- Lobachevsky function (J), 392–393, 401–402, 405
- Loops, 385, 387–388. *See also* Modular loops; Moufang loops
- Lorentz group, 152
- Lorentzian 18, 133–134, 151, 322–329
- Loxodromic homographies, 193
- Lunes, 76, 81
- Manifolds, of constant curvature, 71
- Markovian, 150
- Matrix groups, 331
- Matrix transposition, 44, 213, 406–407
- Maximal sets, of integers, 379
- Median hyperplane, 43, 59, 61, 64
- Medians, of triangles, 117
- Menelaus's Theorem, 115–117
- Meridian(s), 71, 72, 114, 116, 125
- Metalinear, 125–130
- Metric properties, of horocycles, 84–85
- Metrics, xii, 3, 27, 54, 62, 65–67, 134, 137, 157, 322
- Metric spaces, i, xii, 13–18 54–55, 60–61, 74–81, 87, 159–161, 224, 226–228, 390
- Minkowski space, 18
- Minor Theorem of Desargues, 31–32, 37
- Mirrors, 20–21, 23, 118–122, 127, 140, 145–146, 153–155, 189–190, 205–207, 210–211, 228–230, 246, 258, 278, 326
- Möbius group (SL^2), 198
- Möbius n -sphere (I^n), 18, 58, 60–62, 157, 183, 197, 199–201, 218
- Möbius simplex(es), 246–247, 253, 289–391,
- Möbius transformations, 192, 377, 395
- Modular groups, i, xiv, 12, 330–388
- Modular loops, 388, 349, 387–388

- Monon, 224, 226–227, 231, 410
- Mostow Rigidity Theorem, 313
- Moufang loops, 378, 382–384, 385, 387
- Moufang planes ($\mathbb{O}\mathbb{P}_n$), 31, 32, 38
- Multiplicative semigroups, 6, 89–90
- Negative curvature, 71
- Net(s) of rationality, 35–36, 38, 410
- n -gons, 28–29, 313
- n -manifolds, 13–14, 71, 135
- Nonabelian groups, 228
- Nondegenerate conics, real affine plane and, 116, 117
- Non-Desarguesian projective planes, 37–38, 39
- Non-Euclidean geometries, xiii, 1–4, 55–56, 80. *See also* Hyperbolic geometry (H^n); Projective geometry (P^n);
- Non-Euclidean isometries, classification of, 22–24
- Norm, 212, 219, 333, 337, 342, 355–356, 360–361, 364, 368, 372, 378–379
- Normal subgroups, 4–7, 111–112, 95
- n -polytopes, 224–225, 252
- n -sphere(s) (S^n), 23, 60–61, 152, 227
- Null pairs, 133
- Null planes, 133, 137
- Null polarity, 108–109, 178
- Null systems, 132–134
- Octahedral group, 239, 243–245, 248–249
- Octahedron, 238, 309–310, 338–339, 248–250
- Octavian rings, 381
- Octian group ($(2S_4)$, 351
- Octian ring (\mathbb{H}_{43}), 352, 354
- Octonionic integers, 377–383, 387
- Octonionic modular loops, 349, 387–388
- Octonions (\mathbb{O}), 38, 220, 349, 378–387
- One-point compactifications, real inversive sphere and, 186
- “One-to-one” transformations, 7, 106
- Open nodes, in Coxeter diagrams, 254–261
- Opposite dilatations, 96, 148–149
- Opposite isometries, 19–21, 145, 147, 153–155, 190, 199, 217
- Orbifolds, 244
- Ordinary hyperplane(s) ($h\mathbb{P}^{n-1}$). *See also* Ordinary k -planes ($h\mathbb{P}^k$), 17, 48, 51–52, 161
- Ordinary hyperspheres, 62
- Ordinary points, 17, 47–48, 51, 56, 62, 75, 85, 114, 160–161, 184–187, 191, 224, 226, 410
- Orientability, 19–20, 103
- Oriental spaces, 19–24
- Orthogonal duplex group ($O_{[2]n}$), 203–204, 214–215
- Orthogonal group(s) (O_n), 19, 120, 138–139, 141, 144, 155–156, 215, 354, 390, 406, 409
- Orthogonal matrices, 135, 139–144, 147, 203–204, 236
- Orthogonal quadruplex group ($O_{[4]n}$), 203, 214–215
- Orthogonal reflection matrix (matrices), 140, 145, 147, 228, 232, 234, 242–244
- Orthogonal rotation matrix, 140, 146, 171, 228, 234, 242–244
- Orthogonal transformation(s), 139, 143–144, 203–204
- Ortholinear transformations, 117–121, 128–130
- Orthopetic groups (\tilde{O}_n), 149, 409
- Orthopetic transformations, 147
- Ortho- p -gonal groups ($[2, p^+]$), 233–234, 238, 245, 257
- Orthoplex(es) (β_n), 250–252, 381, 410
- Orthorectangular group ($[2, 2]$), 232, 242, 256, 259, 300
- Orthorhombic groups ($[2, 2^+], [2^+, 2]$), 232–233, 256
- Orthoscheme(s), 247, 251, 292–293, 300, 305–307, 309, 313–315, 318, 321, 393

- Oval antiquadratics, 158, 160–161, 164–165
- Oval quadratics (Φ), 45, 54, 58
- Pappian planes, 34, 38, 42–43
- Pappus configuration, 34, 101
- Pappus's Theorem, 34, 37, 42, 102
- Para-apeiral group ($[2, \infty^+]$), 264–265, 266
- Parabolas, 45, 116, 125, 131
- Parabolic coordinates, 185–187
- Parabolic isometries, 189–190
- Parabolic n -sphere (\dot{S}^n), 183–186, 189–190, 194–196
- Parabolic reflection matrices (P_c), 189–190
- Parabolic rotation(s), 125, 130, 131
- Parabolic similarities, 188–189, 194
- Parabolic translation matrices, 189
- Paraclinic group, 267
- Paracompact Coxeter groups, 328, 330, 332, 356–357, 376, 392–393
- Paracompact groups, 304, 309–313, 321, 323–328, 332
- Paracompact hyperbolic groups, 299, 305, 310–311, 313–321, 367, 391
- Paralinear transformations, 121–125, 130
- Parallel hyperplanes, 21, 51–52, 69, 154, 160–161, 322, 407
- Parallelism, 1–3, 27, 52–53, 92
- Parallel lines, 14–16, 70, 85, 114–116, 118, 123, 131, 278, 300–301
- Parallel mirrors, 21, 147
- Parallelograms, 2, 237, 278
- Parallel postulate, 1–3, 15. *See also* Euclid's Fifth Postulate
- Parallel projection, 322–323
- Para- p -gonal group ($[2, p]^+$), 233, 235–236, 245
- Pararhombic group ($[2, 2]^+$), 232, 233, 236, 241, 259
- Parataxes, 116, 117, 124, 131
- Partially ordered sets, polytopes as, 224, 225
- Partitions, 223, 225–226, 229, 253–254, 410
- Pentachoron (pentachora; $\{3, 3, 3\}$), 291–292, 249
- Pericycles, 53–55, 63, 83–86
- Perispheres (S^2), 54–55, 71
- Permutations, 5, 11, 46
- Perpendicularity, 1–3, 27. *See also* Orthogonal entries
- Perspective collineations, 43, 105, 125
- Perspective triangles, 31–34, 42–43
- Perspectivity, 100
- Petrie rotation, 239
- p -gons ($\{p\}$), 226, 231, 237
- Picard group, 337, 376–377
- Plagioschemes, 250–251, 293, 307, 314–316
- Plane symmetry groups, 263–278
- Poincaré model, of hyperbolic plane, 67, 198–199
- Points at infinity, 2, 15, 49, 114, 185–186, 194–196, 229, 283–284
- Polar coordinates, in spherical and pseudospherical geometry, 71–73, 74
- Polarity (polarities), 9, 17–18, 25, 44–47, 50–51, 57–58, 72–73, 108–110, 134, 178
- Pole(s), 20, 44–45, 59
- Polychoric symmetry groups ($[p, q, r]^+$), 291–292, 410
- Polygonal symmetry groups ($[p]^+$), 226–230, 254–256, 290, 410. *See also* p -gonal group(s) ($[p]^+$)
- Polyhedra (polyhedron), 223, 224, 231–246, 238–246, 311–313, 410
- Polyhedral symmetry groups ($[p, q]^+$), 238–246, 410. *See also* Pyritohedral group ($[3^+, 4]$)
- Polytope(s), 80, 223–226, 299, 307–308, 313, 406–407, 410
- Positive definite quadratic forms, 46–47, 49
- Positive scalar group (GZ^+), 90, 104–105, 111–112, 409

- Primitive Hamilton integers (C_3), 356
- Primitive hybrid integers ($A_2 \oplus A_1$), 361–362
- Primitive quaternions, 355, 356
- Prismatic groups, 255–257, 260, 263
- Prismatic honeycombs, 294, 381–384, 410, 285–288
- Projective coordinates, 60, 62, 184–185
- Projective general linear group(s) (PGL_n), 11–12, 104–105, 107–108, 156, 180, 199, 409
- Projective geometry, i, xi, xiii, 4, 27–46, 57–59, 86, 102, 395
- Projective n -space (\mathbb{P}^n , P^n), xii, 17–18, 27, 39–47, 49–51, 87, 105, 152, 395, 407
- Projective orthogonal group(s) (PO_n , PO_2), 19, 120, 138, 142, 149, 156, 409
- Projective pseudo-orthogonal group(s) ($PO_{n-1,1}$, $PO_{1,1}$, $PO_{2,1}$), 19, 128, 138, 156, 193, 199, 200, 218, 336, 390, 409
- Projective special linear group(s) (PSL_n), 104–105, 156, 199, 332, 409
- Projective symplectic group (PSp_n), 134
- Projectivity (projectivities), 11, 43, 44, 103, 108, 163, 196–199
- Pseudocycles, 14, 21, 53, 54–56, 63, 83–85
- Pseudo-modular groups, 349, 355–364, 377
- Pseudo-orthogonal transformations, 151–152, 203
- Pseudospherical geometry, 57, 61–65, 71–73, 86, 395
- Pseudospherical n -space (\check{S}^n), 62, 71–73, 395, 407. *See also* Hyperbolic n -sphere (\check{S}^n)
- Pseudo-unitary group(s) ($U_{n,1}$), 209–210, 409–410
- Quadratic forms, xii, 46–47, 49, 58, 138, 191–192
- Quadrivalent matrices, 211–212, 220
- Quadruplex matrices, quaternions and, 212–213
- Quarter-hypercubic tessellation ($k\delta_{n+1}$), 293–294
- Quaternionic general linear group ($GL_n(\mathbb{H})$), 212–213
- Quaternionic groups, 211–215, 387–388
- Quaternionic gyropetic groups ($\mathbb{H}Gp_n$), 136, 215, 409
- Quaternionic modular groups, 349–388
- Quaternionic symplectic matrices, 203, 213–214, 215
- Quaternions (\mathbb{H}), xii, 12, 38, 136, 203, 211–222, 349, 355–356, 360–361, 372, 378–379, 383–386, 390, 407
- Rational modular group, 330–333, 376
- Real affine n -space (A^n , $\mathbb{R}A^n$), 91, 133, 135, 407
- Real gyropetic groups (Gp_n), 132
- Real inversive n -space (I^n), 390, 407
- Real inversive n -sphere ($\mathbb{R}I^n$), 58, 183
- Rectangular group (D_2 , [2]), 227, 231, 233, 248, 254–255, 290
- Reference frames (Δ), 96–99, 102–103, 106–107, 109–110, 114–116
- Reflection groups, 227–228, 300–308, 410
- Regular honeycombs, 225–226, 292–295, 318, 352, 356–357, 369–371, 376–377, 379–380
- Regular hyperbolic tessellations, 300–301, 332–333, 336
- Regular n -dimensional simplex(es) (α_n), 250, 251
- Regular n -polytopes, symmetry groups of, 251
- Regular polytopes, 225–226, 247, 249–250, 252
- Reversible matrices, 217, 222, 355–358
- Rhombic group, 227, 231, 233, 254–255
- Riemann sphere (\check{S}^2), 194. *See also* Parabolic sphere (\check{S}^2)
- Rotary reflection(s), 22, 236, 238, 257, 261–262

- Rotation groups, 5, 227–228, 260, 264–266, 275, 332, 336, 350–351, 368, 372, 376–377
- Rotation matrices, 118–121, 145–146, 153–154, 155
- Schläflian (σ), 247–248, 253–254, 295, 298, 391, 396–397, 400–405, 410
- Schläfli matrices (matrix), 410, 247, 391
- Self-dual regular cellulations, 341, 305–306
- Semiprojective pseudo-unitary group ($\bar{P}U_{n+1,1}$), 201–202
- Semiquadratic modular groups ($PSL_{1+1}(\mathbb{Z}[\tau])$; $PSL_{1+1}(\mathbb{Z}[\sqrt{d}])$), 335–336, 376
- Semiquadratic unit linear group ($SL_{1+1}(\mathbb{Z}[\sqrt{d}])$), 334
- Semispecial linear groups ($S_2 L_2(\mathbb{E})$), 337, 342, 345
- Separation, 10, 35–36, 81, 410
- Sesquilinear forms, 159–164, 204, 209, 408
- Similarities, 18–20, 106, 138, 147–149, 183, 188–190, 208, 280–283, 286–288, 390
- Simple Lie groups, xiii, 248, 296, 298
- Simply connected spaces, 13
- Skew-fields, 12, 37–41, 90, 101, 365, 369, 373
- Skew-symmetric bilinear forms, 45, 132, 135
- Skew-symmetric matrices, 44, 109, 110, 132, 135, 354
- Space at infinity, Lorentzian, 322–323
- Span, 88, 224
- Special affine group(s) (SA_n), 94, 111–112, 130, 133, 409; Special inversive group (SI_2), 193, 198, 217; Special linear groups (SL_2 ; SL_n), 90, 104–105, 111–112, 130, 132, 213, 330–332, 337, 340, 342–343, 346, 364–368, 372, 409; Special orthogonal group (O_2^+), 120, 139, 141, 180, 409; Special projective group (SP_{n-1}), 104–105; Special scalar groups (SZ_n), 104–105, 337, 342–343, 355–356, 409; Special unitary group(s) (SU_n), 204, 409
- Spherical coordinates, 66–67, 72; Spherical Coxeter groups, xiii, 223, 246–254, 299, 324–325, 327; Spherical geometry, i, 15, 57, 59–61, 71–73, 86, 395; Spherical groups, 296–297, 300, 305–306, 391, 396–397; Spherical tessellations, 238–239, 304; Spherical triangles, 75–76, 80, 254; Spherical unitary n -space (SU^n), 24, 26, 164, 407. *See also* Spherical unitary space
- Spinal $2n$ -spheres ($IU^{[2n-1]/2}$), 201
- Stereographic projection, 195
- Striations, 21–24, 153–156, 229, 332, 337, 340, 343, 356, 359, 360, 362–363, 365, 368–369, 372
- Subloops, of modular loops, 387–388
- Subsidiary i -faces, 224
- Sum operation, with polytopes, 251–252, 410
- Supergroups, 253, 255, 311–312
- Supplementary arcs, 74–75
- Symmetric bilinear forms, 45
- Symmetric groups (S_n), 5
- Symmetric matrices, 44
- Symmetry groups, xiii, 5, 12, 223–271, 292–295, 332, 338–339, 343–344, 356–357, 362, 364–365, 368–371, 373, 376–378
- Symplectic geometry, 132–137
- Symplectic group(s) (Sp_n), 113, 132–137, 214–215, 354, 390, 409
- Symplectic transformations, 132–133, 136, 213–214, 204
- Tangent circles, 62, 192, 193
- Tangent hyperplanes, to n -quadrics, 184
- Tangent hyperspheres, 58–59, 62–63
- Ternispecial linear group ($S_3 L_2(\mathbb{E})$), of Eisenstein integer matrices, 342
- Tesseract ($\{4, 3, 3\}; \gamma_n$), 249. *See also* n -dimensional orthotopes;

- Orthotope(s) (γ_n); Regular n -orthotope(s) (γ_n)
- Tesseractic honeycomb ($\{4, 3, 3, 4\}$), 292, 352, 365, 367
- Tetrads, 29, 31, 33, 41–42, 105
- Tetrahedral group $[3, 3]^+$, 239, 242–245, 248–249, 258–259
- Tilings. *See* Cellulations; Honeycombs; n -honeycombs; Space-fillings; Tessellations; Uniform honeycombs; Uniform tilings
- Timelike vectors/covectors, in Lorentzian ($n+1$)-space, 322–323
- Torohedral groups, 278–289
- Toroidal tessellations, 278–283
- Torus ($S^1 \times S^1$), lattice points on, 279–279
- Totally ordered sets, 225–226
- Transantilinear transformations, 174–175
- Transformations, xiii, 5–9, 11–12, 18–24, 43–46, 59–60, 87–94, 117–125, 125–130, 132, 136, 214, 139, 143–152, 147–148, 156–157, 175–180, 207–211, 216, 330–348, 408
- Transition matrices, 93, 95–96
- Translation(s), 19, 21, 24, 111–112, 116–117, 124, 128, 130–131, 143–144, 147–148, 153–156, 229–230, 263–269, 278, 288–289, 408
- Translation group(s) (T_n), 19, 88, 95, 133, 138, 147, 229, 390, 406, 409
- Translinear transformations, 91–93, 116–117, 172, 175
- Transorthogonal transformation, 143–144
- Transorthoptic transformations, 147–148
- Transpose(s) (transposition), 44, 107–108, 213, 406–407
- Transscalar transformation, 94
- Transunimodular transformations, 130
- Transunitary transformations, 203, 206–209
- Transvections, 123
- Transzygopic transformations, 208
- Triangle Inequality, 55, 78–79, 81
- Triangular tessellation ($\{3, 6\}$), symmetry group of, 271–272, 277–278
- Truncated simplexes, as hyperbolic group fundamental regions, 308
- Twisted areas, 113, 132–134, 137
- Ultra-infinite points, of complete hyperbolic n -space, 17
- Ultraparallel lines, in a hyperbolic plane, 14
- Unicuspal simplexes, as fundamental regions, 325, 327–328
- Uniform honeycombs, 251, 352, 373, 381, 383–384, 386
- Uniform prismatic cellulations, primitive hybrid integers and, 362
- Uniform tilings, 277–278
- Unimodular transformations, 130
- Unitary geometries, 157, 199–202
- Unitary group(s) (U_n), 204–205, 208, 214–215, 406, 409–410
- Unitary inversions, 26, 157, 163, 165
- Unitary metric spaces, 24–26, 159–161
- Unitary spaces, i, 13, 24–26, 159, 203–206.
- See also* Unitary n -space(s)
- Unitary transformations, 203, 206
- Unit dilative group(s) ($\bar{S}D_n$), 95, 111–112, 147–148, 180–182, 409
- Unit hyperchain, 164
- Unit quaternions, 212, 214, 216, 221, 350–352
- Unit scalar group ($\bar{S}Z$), 19, 90, 104–105, 111–112, 120, 141, 152, 176, 180–182, 409
- Upper half-space model, of hyperbolic n -space, 68
- Vector spaces ($\mathbb{F}^n; \mathbf{V}$), xii, xiv, 7–9, 11, 19, 40, 48, 87–91, 168–169, 171–174, 203, 211, 220, 356, 365, 369, 373, 378–379, 383, 408

438

Index

- Vertex figure(s), 226, 237, 356, 381
Vertices, 28–29, 31, 34, 39, 75–77,
100–101, 224, 226, 234, 237
Vinberg polytopes, 305, 308
Wachter’s Theorem, 185
Weights, for extended affine plane
coordinates, 115
Weyl groups, 247–248, 295–296. *See also*
Coxeter groups
Witt symbols, 296–297, 391, 398–405, 407
Wreath product(s), 249, 407
 x -axis, 123, 114, 125
 x -intercept, 114, 410
 y -axis, 114, 123, 125
 y -intercept, 113–114, 118, 410
Zygopetic group(s), 209, 409
Zygopetic transformations, 208