Cambridge University Press
978-1-107-10340-5 — Geometries and Transformations
Norman W. Johnson
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Geometries and Transformations

Euclidean and other geometries are distinguished by the transformations that preserve their essential properties. Using linear algebra and transformation groups, this book provides a readable exposition of how these classical geometries are both differentiated and connected. Following Cayley and Klein, the book builds on projective and inversive geometry to construct "linear" and "circular" geometries, including classical real metric spaces like Euclidean, hyperbolic, elliptic, and spherical, as well as their unitary counterparts. The first part of the book deals with the foundations and general properties of the various kinds of geometries. The latter part studies discrete-geometric structures and their symmetries in various spaces. Written for graduate students, the book includes numerous exercises and covers both classical results and new research in the field. An understanding of analytic geometry, linear algebra, and elementary group theory is assumed.

Norman W. Johnson was Professor Emeritus of Mathematics at Wheaton College, Massachusetts. Johnson authored and coauthored numerous journal articles on geometry and algebra, and his 1966 paper 'Convey Polyhedra with Regular Faces' enumerated what have come to be called the Johnson solids. He was a frequent participant in international conferences and a member of the American Mathematical Society and the Mathematical Association of America.

Norman Johnson: An Appreciation
by Thomas Banchoff

Norman Johnson passed away on July 13, 2017, just a few months short of the scheduled appearance of his magnum opus, *Geometries and Transformations*. It was just over fifty years ago in 1966 that his most famous publication appeared in the *Canadian Journal of Mathematics*: "Convex Polyhedra with Regular Faces," presenting the complete list of ninety-two examples of what are now universally known as Johnson Solids. He not only described all of the symmetry groups of the solids; he also devised names for each of them, containing enough information to exhibit their constituent elements and the way they were arranged.

Now, as a result of this new volume, he will be further appreciated for his compendium of all the major geometric theories together with their full groups of transformations that preserve their essential characteristic features. In this process, he follows the spirit of Felix Klein and his Erlangen Program, first enunciated in 1870. Once again, a specific personal contribution to this achievement is his facility for providing descriptive names for each of the types of transformations that characterize a given geometry. His book will be an instant classic, giving an overview that will serve as an introduction to new subjects as well as an authoritative reference for established theories. In the last few chapters, he includes a similar treatment of finite symmetry groups in the tradition of his advisor and mentor, H. S. M. Coxeter.

It is fitting that the cover image and the formal announcement of the forthcoming book were first released by Cambridge University Press at the annual summer MathFest of the Mathematical Association of America. Norman Johnson was a regular participant in the MathFest meetings, and in 2016 he was honored as the person there who had been a member of the MAA for the longest time, at sixty-three years. His friends and colleagues are saddened to learn of his passing even as we look forward to his new geometrical masterwork.

GEOMETRIES AND TRANSFORMATIONS

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PREFACE

The plane and solid geometry of Euclid, long thought to be the only kind there was or could be, eventually spawned both higher-dimensional Euclidean space and the classical non-Euclidean metric spaces—hyperbolic, elliptic, and spherical—as well as more loosely structured systems such as affine, projective, and inversive geometry. This book is concerned with how these various geometries are related, with particular attention paid to their transformation groups, both continuous and discrete. In the spirit of Cayley and Klein, all the systems to be considered will be presented as specializations of some projective space.

As first demonstrated by von Staudt in 1857, projective spaces have an intrinsic algebraic structure, which is manifested when they are suitably coordinatized. While synthetic methods can still be employed, geometric results can also be obtained via the underlying algebra. In a brief treatment of the foundations of projective geometry, I give a categorical set of axioms for the real projective plane. In developing the general theory of projective n-space and other spaces, however, I make free use of coordinates, mappings, and other algebraic concepts, including cross ratios.

Like Euclidean space, elliptic and hyperbolic spaces are both "linear," in the sense that two coplanar lines meet in at most one point. But spherical space is "circular": two great circles on a sphere always meet in a pair of antipodal points. Each of these spaces can be coordinatized over the real field, and each of them can be derived directly or indirectly from real projective space. For the three linear geometries, a metric can be induced by specializing an *absolute polarity* in projective n-space P^n or one of its hyperplanes. Fixing a *central inversion* in inversive n-space, realized as an oval n-quadric in P^{n+1}, similarly produces one or another kind of circular metric space.

Each of these real metric spaces has a *unitary* counterpart that can be derived analogously from complex projective space and coordinatized over the complex field. A unitary n-space may be represented isometrically by a real space of dimension $2n$ or $2n + 1$, and vice versa—a familiar example being the Argand diagram identifying complex numbers with points of the Euclidean plane. Many of the connections between real spaces and representations involving complex numbers or quaternions are well known, but new ground has been broken by Ahlfors (1985), Wilker (1993), Goldman (1999), and others.

From the point and hyperplane coordinates of a projective space we may construct coordinates for a metric space derived from it, in terms of which properties of the space may be described and distances and angles measured. Many of the formulas obtained exhibit an obvious symmetry, a consequence of the Principle of Duality that permits the interchange of projective subspaces of complementary dimensions. Along with the related concept of dual vector spaces, that principle also finds expression in convenient notations for coordinates and for corresponding bilinear and quadratic forms. The pseudo-symmetry of *real* and *imaginary* is likewise evident in inversive geometry as well as in the metric properties of hyperbolic and elliptic geometry.

With coordinates taken as vectors, transformations may be represented by matrices, facilitating the description and classification of geometric groups, as in the famous *Erlanger Programm* proposed by

Klein in 1872. Many of these occur among what Weyl called the "classical groups," and each geometric group is placed in an algebraic context, so that its relationship to other groups may be readily seen.

While there are still excellent treatments like that of Sossinsky (2012) that make little use of vectors and matrices, in recent years the study of geometries and transformations has benefited from the application of the methods of linear algebra. The treatment here has much in common with the approaches taken by such writers as Artzy (1965), Snapper & Troyer (1971), Burn (1985), and Neumann, Stoy & Thompson (1994). Conversely, as shown in some detail by Beardon (2005), many algebraic concepts can be illuminated by placing them in the framework of geometric transformations.

On the other hand, this book is not meant to be a text for a course in linear algebra, projective geometry, or non-Euclidean geometry, nor is it an attempt to synthesize some combination of those subjects in the manner of Baer (1952) or Onishchik & Sulanke (2006). My objective is rather to explain how all these things fit together.

The first ten chapters of the book essentially deal with transformations of geometries, and the last five with symmetries of geometric figures. Each continuous group of geometric transformations has various discrete subgroups, which are of interest both in their own right and as the symmetry groups of polytopes, tilings, or patterns. Every isometry of spherical, Euclidean, or hyperbolic space can be expressed as the product of reflections, and discrete groups generated by reflections—known as *Coxeter groups*—are of particular importance. Finite and infinite symmetry groups are subgroups or extensions of spherical and Euclidean Coxeter groups. Irreducible Euclidean Coxeter groups are closely related to simple Lie groups.

Certain numerical properties of hyperbolic Coxeter groups have been determined by Ruth Kellerhals and by John G. Ratcliffe and Steven Tschantz, as well as by the four of us together. A number of discrete groups operating in hyperbolic n-space ($2 \leq n \leq 5$) correspond to linear fractional transformations over rings of real, complex, or quaternionic integers. The associated *modular groups* are discussed in the final two chapters, which have for the most part been extracted from work done jointly with Asia Ivić Weiss.

One of the advantages of building a theory of geometries and transformations on the foundation of vector spaces and matrices is that many geometric propositions are easy to verify algebraically. Consequently, many proofs are only sketched or left as exercises. Where appropriate, however, I provide explanatory details or give references to places where proofs can be found. These include both original sources and standard texts like those by Veblen & Young and Coxeter, as well as relevant works by contemporary authors. In any case, results are not generally presented as formal theorems.

The reader is assumed to have a basic understanding of analytic geometry and linear algebra, as well as of elementary group theory. I also take for granted the reader's familiarity with the essentials of Euclidean plane geometry. Some prior knowledge of the properties of non-Euclidean spaces and of real projective, affine, and inversive geometry would be helpful but is not required. A few Preliminaries provide a short review. It is my hope that the book may serve not only as a possible text for an advanced geometry course but also as a readable exposition of how the classical geometries are both differentiated and connected by their groups of transformations.

Many of the ideas in this book are variations on themes pursued by my mentor, Donald Coxeter, who in the course of his long life did much to rescue geometry from oblivion. I am also grateful for the support and encouragement of my fellow Coxeter students Asia Weiss, Barry Monson, and the late John Wilker, as well as many stimulating exchanges with Egon Schulte and Peter McMullen. It was also a privilege to collaborate with Ruth Kellerhals, John Ratcliffe, and Steve Tschantz. Many details of the book itself benefited from the assistance

of Tom Ruen, whose comments and suggestions were extremely helpful, along with Anton Sherwood and George Olshevsky. And I owe many thanks to my editors and the production staff at Cambridge, especially Katie Leach, Adam Kratoska, and Mark Fox.