PRIME NUMBERS AND THE RIEMANN HYPOTHESIS

Prime numbers are beautiful, mysterious, and beguiling mathematical objects. The mathematician Bernhard Riemann made a celebrated conjecture about primes in 1859, the so-called Riemann Hypothesis, which remains to be one of the most important unsolved problems in mathematics. Through the deep insights of the authors, this book introduces primes and explains the Riemann Hypothesis.

Students with minimal mathematical background and scholars alike will enjoy this comprehensive discussion of primes. The first part of the book will inspire the curiosity of a general reader with an accessible explanation of the key ideas. The exposition of these ideas is generously illuminated by computational graphics that exhibit the key concepts and phenomena in enticing detail. Readers with more mathematical experience will then go deeper into the structure of primes and see how the Riemann Hypothesis relates to Fourier analysis using the vocabulary of spectra. Readers with a strong mathematical background will be able to connect these ideas to historical formulations of the Riemann Hypothesis.

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Preface

The Riemann Hypothesis is one of the great unsolved problems of mathematics, and the reward of $1,000,000 of *Clay Mathematics Institute* prize money awaits the person who solves it. But – with or without money – its resolution is crucial for our understanding of the nature of numbers.

There are several full-length books recently published, written for a general audience, that have the Riemann Hypothesis as their main topic. A reader of these books will get a fairly rich picture of the personalities engaged in the pursuit, and of related mathematical and historical issues.[1]

This is *not* the mission of the book that you now hold in your hands. We aim – instead – to explain, in as direct a manner as possible and with the least mathematical background required, what this problem is all about and why it is so important. For even before anyone proves this *hypothesis* to be true (or false!), just getting familiar with it, and with some of the ideas behind it, is exciting. Moreover, this hypothesis is of crucial importance in a wide range of mathematical fields; for example, it is a confidence-booster for computational mathematics: even if the Riemann Hypothesis is never proved, assuming its truth (and that of closely related hypotheses) gives us an excellent sense of how long certain computer programs will take to run, which, in some cases, gives us the assurance we need to initiate a computation that might take weeks or even months to complete.

Here is how the Princeton mathematician Peter Sarnak describes the broad impact the Riemann Hypothesis has had[2]:

> "The Riemann hypothesis is the central problem and it implies many, many things. One thing that makes it rather unusual in mathematics today is that there must be over five hundred papers – somebody should

[1] See, e.g., *The Music of the Primes* by Marcus du Sautoy (2003) and *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics* by John Derbyshire (2003).

[2] See page 222 of *The Riemann hypothesis: the greatest unsolved problem in mathematics* by Karl Sabbagh (2002).

Figure 0.1. Peter Sarnak. Photo by William Stein

> go and count – which start 'Assume the Riemann hypothesis[3],' and the conclusion is fantastic. And those [conclusions] would then become theorems . . . With this one solution you would have proven five hundred theorems or more at once."

So, what *is* the Riemann Hypothesis? Below is a *first description* of what it is about. The task of our book is to develop the following boxed paragraph into a fuller explanation and to convince you of the importance and beauty of the mathematics it represents. We will be offering, throughout our book, a number of different – but equivalent – ways of precisely formulating this hypothesis (we display these in boxes). When we say that two mathematical statements are "equivalent," we mean that, given the present state of mathematical knowledge, we can prove that if either one of those statements is true, then the other is true. The endnotes will guide the reader to the relevant mathematical literature.

What sort of Hypothesis is the Riemann Hypothesis?

Consider the seemingly innocuous series of questions:

- How many prime numbers (2, 3, 5, 7, 11, 13, 17, . . .) are there less than 100?
- How many less than 10,000?
- How many less than 1,000,000?

More generally, how many primes are there less than any given number X?

[3] Technically, a generalized version of the Riemann hypothesis (see Chapter 38 below).

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Riemann proposed, a century and half ago, a strikingly simple-to-describe "very good approximation" to the number of primes less than or equal to a given number X. We now see that if we could prove this *Hypothesis of Riemann* we would have the key to a wealth of powerful mathematics. Mathematicians are eager to find that key.

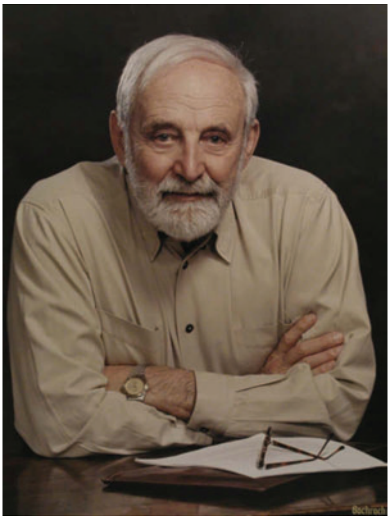

Figure 0.2. Raoul Bott. Courtesy of the Department of Mathematics, Harvard University

The mathematician Raoul Bott – in giving advice to a student – once said that whenever one reads a mathematics book or article, or goes to a math lecture, one should aim to come home with something very specific (it can be small, but should be *specific*) that has application to a wider class of mathematical problems than was the focus of the text or lecture. If we were to suggest some possible *specific* items to come home with, after reading our book, three key phrases – **prime numbers**, **square-root accurate**, and **spectrum** – would head the list. As for words of encouragement to think hard about the first of these, i.e., prime numbers, we can do no better than to quote a paragraph of Don Zagier's classic 12-page exposition, *The First 50 Million Prime Numbers*:

> "There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, [they are] the most arbitrary and ornery objects studied by mathematicians: they grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the

Figure 0.3. Don Zagier. Photo by William Stein

> prime numbers exhibit stunning regularity, that there are laws governing their behavior, and that they obey these laws with almost military precision."

Mathematics is flourishing. Each year sees new exciting initiatives that extend and sharpen the applications of our subject, new directions for deep exploration – and finer understanding – of classical as well as very contemporary mathematical domains. We are aided in such explorations by the development of more and more powerful tools. We see resolutions of centrally important questions. And through all of this, we are treated to surprises and dramatic changes of viewpoint; in short: marvels.

And what an array of wonderful techniques allow mathematicians to do their work: framing *definitions;* producing *constructions;* formulating *analogies relating disparate concepts, and disparate mathematical fields*; posing *conjectures,* that cleanly shape a possible way forward; and, the keystone: providing unassailable *proofs* of what is asserted, the idea of doing such a thing being itself one of the great glories of mathematics.

Number theory has its share of this bounty. Along with all these modes of theoretical work, number theory also offers the pure joy of numerical experimentation, which – when it is going well – allows you to witness the intricacy of numbers and profound inter-relations that cry out for explanation. It is striking how little you actually have to know in order to appreciate the revelations offered by numerical exploration.

Our book is meant to be an introduction to these pleasures. We take an experimental view of the fundamental ideas of the subject buttressed by numerical computations, often displayed as graphs. As a result, our book is profusely illustrated, containing 131 figures, diagrams, and pictures that accompany the text.[4]

There are few mathematical equations in Part I. This first portion of our book is intended for readers who are generally interested in, or curious about, mathematical ideas, but who may not have studied any advanced topics. Part I is devoted to conveying the essence of the Riemann Hypothesis and explaining why it is so intensely pursued. It requires a minimum of mathematical knowledge, and does not, for example, use calculus, although it would be helpful to know – or to learn on the run – the meaning of the concept of *function*. Given its mission, Part I is meant to be complete, in that it has a beginning, middle, and end. We hope that our readers who only read Part I will have enjoyed the excitement of this important piece of mathematics.

Part II is for readers who have taken at least one class in calculus, possibly a long time ago. It is meant as a general preparation for the type of Fourier analysis that will occur in the later parts. The notion of spectrum is key.

Part III is for readers who wish to see, more vividly, the link between the placement of prime numbers and (what we call there) the *Riemann spectrum*.

Part IV requires some familiarity with complex analytic functions, and returns to Riemann's original viewpoint. In particular it relates the "Riemann spectrum" that we discuss in Part III to the *nontrivial zeroes of the Riemann zeta function*. We also provide a brief sketch of the more standard route taken by published expositions of the Riemann Hypothesis.

The end-notes are meant to link the text to references, but also to provide more technical commentary with an increasing dependence on mathematical background in the later chapters. References to the end notes will be in brackets.

We wrote our book over the past decade, but devoted only one week to it each year (a week in August). At the end of our work-week for the book, each year, we put our draft (mistakes and all) on line to get response from readers.[5] We therefore accumulated much important feedback, corrections, and requests from readers, especially J. S. Markovitch who very carefully proofread the final draft.[6] We thank them infinitely.

[4] We created the figures using the free SageMath software (see `http://www.sagemath.org`). Complete source code is available, which can be used to recreate every diagram in this book (see `http://wstein.org/rh`). More adventurous readers can try to experiment with the parameters for the ranges of data illustrated, so as to get an even more vivid sense of how the numbers "behave." We hope that readers become inspired to carry out numerical experimentation, which is becoming easier as mathematical software advances.

[5] See `http://library.fora.tv/2014/04/25/Riemann_Hypothesis_The_Million_Dollar_Challenge` which is a lecture – and Q & A – about the composition of this book.

[6] Including Dan Asimov, Bret Benesh, Keren Binyaminov, Harald Bögeholz, Louis-Philippe Chiasson, Keith Conrad, Karl-Dieter Crisman, Nicola Dunn, Thomas Egense, Bill Gosper, Andrew Granville, Shaun Griffith, Michael J. Gruber, Robert Harron, William R. Hearst III, David Jao, Fredrik Johansson, Jim Markovitch, David Mumford, James Propp, Andrew Solomon, Dennis Stein, and Chris Swenson.