MATHEMATICS OF THE BOND MARKET

Mathematical models of bond markets are of interest to researchers working in applied mathematics, especially in mathematical finance. This book concerns bond market models in which random elements are represented by Lévy processes. These are more flexible than classical models and are well suited to describing prices quoted in a discontinuous fashion.

The book's key aims are to characterize bond markets that are free of arbitrage and to analyze their completeness. Nonlinear stochastic partial differential equations (SPDEs) are an important tool in the analysis. The authors begin with a relatively elementary analysis in discrete time, suitable for readers who are not familiar with finance or continuous time stochastic analysis. The book should be of interest to mathematicians, in particular to probabilists, who wish to learn the theory of the bond market and to be exposed to attractive open mathematical problems.

Encyclopedia of Mathematics and Its Applications

This series is devoted to significant topics or themes that have wide application in mathematics or mathematical science and for which a detailed development of the abstract theory is less important than a thorough and concrete exploration of the implications and applications.

Books in the **Encyclopedia of Mathematics and Its Applications** cover their subjects comprehensively. Less important results may be summarized as exercises at the ends of chapters. For technicalities, readers can be referred to the bibliography, which is expected to be comprehensive. As a result, volumes are encyclopedic references or manageable guides to major subjects.

Encyclopedia of Mathematics and Its Applications

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit www.cambridge.org/mathematics.

124 F. W. King *Hilbert Transforms I*
125 F. W. King *Hilbert Transforms II*
126 O. Calin and D.-C. Chang *Sub-Riemannian Geometry*
127 M. Grabisch *et al*. *Aggregation Functions*
128 L. W. Beineke and R. J. Wilson (eds.) with J. L. Gross and T. W. Tucker *Topics in Topological Graph Theory*
129 J. Berstel, D. Perrin and C. Reutenauer *Codes and Automata*
130 T. G. Faticoni *Modules over Endomorphism Rings*
131 H. Morimoto *Stochastic Control and Mathematical Modeling*
132 G. Schmidt *Relational Mathematics*
133 P. Kornerup and D. W. Matula *Finite Precision Number Systems and Arithmetic*
134 Y. Crama and P. L. Hammer (eds.) *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*
135 V. Berthé and M. Rigo (eds.) *Combinatorics, Automata and Number Theory*
136 A. Kristály, V. D. Rădulescu and C. Varga *Variational Principles in Mathematical Physics, Geometry, and Economics*
137 J. Berstel and C. Reutenauer *Noncommutative Rational Series with Applications*
138 B. Courcelle and J. Engelfriet *Graph Structure and Monadic Second-Order Logic*
139 M. Fiedler *Matrices and Graphs in Geometry*
140 N. Vakil *Real Analysis through Modern Infinitesimals*
141 R. B. Paris *Hadamard Expansions and Hyperasymptotic Evaluation*
142 Y. Crama and P. L. Hammer *Boolean Functions*
143 A. Arapostathis, V. S. Borkar and M. K. Ghosh *Ergodic Control of Diffusion Processes*
144 N. Caspard, B. Leclerc and B. Monjardet *Finite Ordered Sets*
145 D. Z. Arov and H. Dym *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*
146 G. Dassios *Ellipsoidal Harmonics*
147 L. W. Beineke and R. J. Wilson (eds.) with O. R. Oellermann *Topics in Structural Graph Theory*
148 L. Berlyand, A. G. Kolpakov and A. Novikov *Introduction to the Network Approximation Method for Materials Modeling*
149 M. Baake and U. Grimm *Aperiodic Order I: A Mathematical Invitation*
150 J. Borwein *et al*. *Lattice Sums Then and Now*
151 R. Schneider *Convex Bodies: The Brunn-Minkowski Theory (Second Edition)*
152 G. Da Prato and J. Zabczyk *Stochastic Equations in Infinite Dimensions (Second Edition)*
153 D. Hofmann, G. J. Seal and W. Tholen (eds.) *Monoidal Topology*
154 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras I: The Vidav–Palmer and Gelfand–Naimark Theorems*
155 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables (Second Edition)*
156 L. W. Beineke and R. J. Wilson (eds.) with B. Toft *Topics in Chromatic Graph Theory*
157 T. Mora *Solving Polynomial Equation Systems III: Algebraic Solving*
158 T. Mora *Solving Polynomial Equation Systems IV: Buchberger Theory and Beyond*
159 V. Berthé and M. Rigo (eds.) *Combinatorics, Words and Symbolic Dynamics*
160 B. Rubin *Introduction to Radon Transforms: With Elements of Fractional Calculus and Harmonic Analysis*
161 M. Ghergu and S. D. Taliaferro *Isolated Singularities in Partial Differential Inequalities*
162 G. Molica Bisci, V. D. Radulescu and R. Servadei *Variational Methods for Nonlocal Fractional Problems*
163 S. Wagon *The Banach–Tarski Paradox (Second Edition)*
164 K. Broughan *Equivalents of the Riemann Hypothesis I: Arithmetic Equivalents*
165 K. Broughan *Equivalents of the Riemann Hypothesis II: Analytic Equivalents*
166 M. Baake and U. Grimm (eds.) *Aperiodic Order II: Crystallography and Almost Periodicity*
167 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras II: Representation Theory and the Zel'manov Approach*
168 A. Yu. Khrennikov, S. V. Kozyrev and W. A. Zúñiga-Galindo *Ultrametric Pseudodifferential Equations and Applications*
169 S. R. Finch *Mathematical Constants II*
170 J. Krajíček *Proof Complexity*
171 D. Bulacu, S. Caenepeel, F. Panaite and F. Van Oystaeyen *Quasi-Hopf Algebras*
172 P. McMullen *Geometric Regular Polytopes*
173 M. Aguiar and S. Mahajan *Bimonoids for Hyperplane Arrangements*

Encyclopedia of Mathematics and Its Applications

Mathematics of the Bond Market

A Lévy Processes Approach

MICHAŁ BARSKI
Faculty of Mathematics, Informatics and Mechanics,
University of Warsaw

JERZY ZABCZYK
Institute of Mathematics,
Polish Academy of Sciences

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107101296
DOI: 10.1017/9781316181836

First published 2020

Printed in the United Kingdom by TJ International Ltd, Padstow Cornwall

A catalogue record for this publication is available from the British Library.

ISBN 978-1-107-10129-6 Hardback

To our wives Anna and Barbara

Contents

Preface

The Field

The book is devoted to the mathematical theory of the bond market, which is a part of mathematical finance. It is addressed to mathematicians, especially to probabilists who are not necessarily familiar with mathematical finance. In fact, Part I – out of the four parts of this book – treats the subject in discrete time and the knowledge of classical probability, as presented in Feller [51], is sufficient for its understanding.

Mathematical finance is today a part of stochastic analysis. Such concepts as stochastic integral and martingales play a fundamental role in finance. For instance, the mathematical theory of stochastic integration is well developed for large classes of integrators and integrands, and general concepts are ideally suited to financial modelling. Integrators are *price processes* of financial commodities, integrands describe *trading strategies* and the integrals represent *accumulated wealth*.

Basic objects of the theory are two random fields $P(t,T), f(t,T)$, $0 \leq t \leq T$, and a stochastic process $R(t), t \geq 0$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. They are related to each other by the formulas

$$P(t,T) = e^{-\int_t^T f(t,s)ds}, \quad 0 \leq t \leq T, \qquad R(t) = f(t,t), \ t \geq 0,$$

and interpreted as, respectively, *bond prices*, *forward rates* and *short rate*. In particular, $P(t,T)$ is the price of a bond at time t that matures at time T, that is, the owner of the bond will receive cash $P(T,T)$ at time T.

The theory is relatively young, approximately 40 years old, and poses new mathematical questions. An important one is about the *absence of arbitrage*. Intuitively, the market should not allow agents to accumulate wealth, by clever investments, without the possibility of facing losses. This property of bond models is mathematically expressed in the concept of non-arbitrage. A related question concerns conditions under which there exists a *martingale measure* for the bond prices, that is, a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that for each $T \geq 0$, the process of discounted bond prices

$$\hat{P}(t,T) = e^{-\int_0^t R(s)ds} P(t,T), \quad t \in [0,T]$$

is a local martingale under $\mathbb{Q}$. Problems of this type have never been asked earlier. Another question is that of *completeness* of the market. Mathematically it is equivalent to the condition that each, say, bounded $\mathcal{F}_{T^*}$-measurable random variable, with $T^* > 0$, can be represented as a sum of a constant and a stochastic integral, over the interval $[0, T^*]$, with integrator $\hat{P}(t,\cdot)$, $t \in [0, T^*]$.

The time evolution of bond prices, short rates and forward rates is studied using the theory of Lévy processes and stochastic differential equations. In fact, applications of the theory of stochastic partial differential equations with Lévy noise – a relatively young branch of stochastic processes – are discussed in the book in great detail.

For the reader's convenience the book starts with an extensive treatment of discrete time models. Here the role of Lévy processes is played by random walks.

Lévy Modelling

A good model of bond prices should satisfy several conditions and allow easy confrontation with reality. Stochastic processes used in applications are numerically "tractable" if they are of Markov type or, more specifically, if they are solutions of stochastic equations. For them, at least theoretically, one can find finite dimensional distributions by solving parabolic equations of Kolmogorov type.

As already mentioned, the book is concerned with models in which random elements are represented through Lévy processes that are natural generalizations of the Wiener process. There are several reasons to go outside the classical paradigm. Models based on Lévy processes allow one to treat situations leading to heavy-tailed distributions. Moreover, they allow exploiting the full strength of Markovian modelling because the most general Markov processes are solutions of stochastic differential equations driven by Lévy processes. Since Lévy processes admit jumps, they are well suited to describing prices quoted on exchanges in a discontinuous fashion.

The mathematical theory of the bond market sets a specific area in financial mathematics. Its analysis involves an infinite dimensional setting because basic objects of the theory, bond prices and forward rates, are function-valued processes. Such a framework can hardly be found in classical stock market models.

The research literature on the Lévy bond market is very extensive and growing with an increasing speed. The starting point was the seminal 1997 papers by Björk, Kabanov and Runggaldier [20] and Björk, Di Masi, Kabanov and Runggaldier [19] that laid down the foundations for the analysis of the bond market in a stochastic model with a general discontinuous noise and prompted further research in that direction. Important contributions describing basic properties of the bond market with Lévy noise are due to Eberlein, Jacod and Raible [48], [47]. Interesting results were published in particular by Filipović, Tappe and Teichmann [52], [54], [56], [57]. Several issues were treated by the authors of the present book [5], [3], [7], [6], [8] and [9] and together with Jakubowski [76], [4]. As the results are mathematically rather involved, it seemed that a book on the subject giving solid foundations for future research would be a welcome contribution.

There are rather few books containing material on Lévy modelling of the financial market and there is none devoted to the bond market. The well-known book by Cont and Tankov [29] deals with stock markets. Only in the final comments does it indicate Lévy bond markets as a possible direction of research. Similarly Applebaum [2] considers some problems of Lévy stock markets limiting his discussion of the bond market to some far-reaching suggestions. In the book [100] by Peszat and Zabczyk a more extensive treatment is available, but many questions were left for further study. The well-known books of Carmona and Tehranchi [25] and Filipović [52] as well as part of the classical monograph of Björk [16] are devoted to the bond market, but all deal with models based on the Wiener process.

Aims of the Book

Our first aim is to mathematically characterize those Lévy bond markets that are free of arbitrage. Intuitively, a market is arbitrage free if a trader is not able to generate profit without taking risk. A sufficient condition for that is the existence of the so-called martingale probability measure equivalent to the basic one.

The second main concept we analyze is completeness of the market. Again, intuitively, a market is complete if a trader can construct a strategy that reproduces any prespecified financial contract.

It turns out that a useful tool to construct arbitrage-free bond market models is provided by stochastic equations. The stochastic equations that appear here are nonlinear and sometimes with partial derivatives. Their analysis is one of the main novelties of the book.

The analysis of the mentioned issues is mathematically rather involved. To make the material more accessible we begin by considering a discrete time setting. It is of independent interest, and almost all results from the continuous time framework are proven here in a more direct way.

Structure of the Book

The book consists of four parts preceded by an Introduction that, in particular, contains some financial background. Part I deals with discrete time models and it is aimed at those readers who have had no previous contact with mathematical finance. The randomness is generated by a sequence of independent identically distributed random variables, a counterpart of the increments of Lévy processes. The results described in this part suggest what can be obtained in the much more challenging continuous time setting. Part II is an overview of results from stochastic analysis required for the continuous time framework. In Part III we treat in detail bond markets driven by Lévy processes, covering such topics as non-arbitrage conditions including the derivation of the general Heath–Jarrow–Morton conditions as well as the existence of martingale measures and completeness of the models. Special attention is paid to the important class of models with affine term structure and general models with Markovian factors. In Part IV we construct arbitrage-free models with the use of stochastic partial differential equations with Lévy noise. The equations that appear there are of unusual type as their coefficients, both linear and nonlinear, are of nonlocal character.

Acknowledgements

It is a pleasure to thank our colleagues Tomas Björk, Nikos Frangos, Jacek Jakubowski, Szymon Peszat, Anna Rusinek and Thorsten Schmidt for discussions on the topics of this book. We also thank Jerzy Trzeciak for language consultations.

The first author thanks Leipzig University and Warsaw University for good working conditions and the Institute of Mathematics Polish Academy of Sciences for constant support. The second author is grateful to his home institution, the Institute of Mathematics Polish Academy of Sciences for providing a stimulating research environment. Financial support from the Warsaw Center of Mathematical and Computer Sciences is gratefully acknowledged.

Any comments and remarks from the readers are welcome and can be sent to mbarski@mimuw.edu.pl.