Introduction

I.1 Bonds

Bonds are financial assets issued by governments, central banks or companies. Their holders receive some fixed payments at future dates. The lifetime of a bond is specified by its *maturity* – the date when the *nominal value* of the bond is paid. All previous payments are called *coupons* and they are usually fixed as fractions of the nominal value of the bond. The payments received by the holder, although fixed, can, however, be influenced by the *credit rating* of the issuer. This means that in case of the issuer's bankruptcy the promised payments can be reduced or even cancelled.

There are many kinds of bonds depending on the length of maturity, the frequency of coupon dates and the credit rating of the issuer. Bonds with maturities between 2 and 5 years are called *short-term bonds* or *bills*, those with maturities between 6 and 12 years are *medium-term bonds* or *notes*. Maturities of the *long-term bonds* exceed 12 years but usually are not longer than 30 years. *Perpetual bonds* called also *consols* have infinite maturities, so they pay a stream of coupons forever. The credit rating of the issuer, which describes his/her *default probability*, is assigned by rating agencies and usually denoted by a combination of letters A, B, C, D corrected by $+$ or $-$. The highest rank AAA is followed by $AA+$, AA and so on till D. Coupons of a bond with a high credit rating offer lower payments than those with a low credit rating but the probability that they will be paid without reduction is higher. Real gamblers who have no risk aversion may invest money in *junk bonds* offering profitable coupons that are, however, biased by a critical rating value. Bonds and related financial contracts constitute an enormous market with trading volume exceeding that of the shares. Instead of going deeper into classifying the variety of bonds, we will now focus on their mathematical description.

In this book we consider *zero coupon risk-free bonds* with nominal value 1, which means that 1 unit of cash is paid to the holder at maturity. There are no coupons and the default probability of the issuer disappears. A bond with maturity $T > 0$ will also be called a T-*bond* as it is uniquely characterized by its maturity, and its price at time

$t \in [0, T]$ will be denoted by $P(t, T)$. So, $P(0, T)$ stands for the initial price of the T-bond and $P(T, T) = 1$ is its nominal value. The set of all maturities will be assumed to be $[0, +\infty)$, and by a *bond market* we mean the family of T-bonds with $T \geq 0$. Our model framework with an infinite number of bonds is a kind of mathematical idealization of the real bond market where only a finite number of bonds are traded, but it can be justified by a huge variety of available bonds. Consideration of bonds without coupons is not really restrictive. In fact, every non-zero coupon bond can be represented as a combination of zero coupon bonds no matter what its coupon scheme. A property that does not feature in our study is the default possibility of the issuer. So, the standing assumption in the whole book is that the nominal value of each bond will be paid with probability one.

The family of prices

$$P(t, T), \quad t \in [0, T]; \qquad T \geq 0$$

is called the *term structure of zero coupon bond prices*. The number $P(t, T)$ can be identified with a *risk-free investment* with two dates of payment given by the pair (t, T), where $0 \leq t < T$. Indeed, buying the T-bond for $P(t, T)$ units of cash at time t provides the payoff $P(T, T) = 1$ at T. Since $P(t, T)$ and the nominal value are known at time t, the deal is free of risk. Although $P(0, T)$ and $P(T, T)$ are known at $t = 0$, the price evolution $t \mapsto P(t, T)$ on $(0, T)$ is random and is affected by the state of the economy. One should realize that bonds and stocks represent two competitive parts of the security market that combine investment gain and risk in a different way. In a good economical situation the stock market is developing well and its low investment risk attracts investors. In this situation the bond market, to be competitive, must offer high gains, i.e. the difference between current prices and nominal values of bonds should be high. This means that bond prices are low. Conversely, high bond prices correspond to high uncertainty on the stock market related to economical perturbations.

I.2 Models

It is of prime importance to develop stochastic models that describe the evolution of bond price processes in a way that reflects their real behaviour. Now we briefly introduce models investigated in the book.

Heath–Jarrow–Morton Models

A forward rate is a random function of two variables

$$f(t, T) = f(\omega, t, T), \qquad t \in [0, T], \quad T \geq 0,$$

such that for each $t \geq 0$ the trajectory $T \mapsto f(t, T)$ is known at time t. The bond prices are then given by

$$P(t, T) = e^{-\int_t^T f(t,u)du}, \quad t \in [0, T], \quad T \geq 0.$$

The previous bond price formula reflects two important properties observed on the real market. The bond price $P(t,T)$ behaves in a regular way in T and is chaotic in t providing that time fluctuations of the forward rate are sufficiently rough. In the seminal paper [67] of Heath, Jarrow and Morton the forward rate dynamics has the form

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t), \qquad t \in [0,T], \quad T \geq 0, \\ f(0,T) &= f_0(T), \quad T \geq 0, \end{aligned} \tag{I.2.1}$$

where W is a Wiener process. In this approach the forward rate is viewed as a family of stochastic processes $t \mapsto f(t,T)$ parametrized by $T \geq 0$. Then (I.2.1) is a system of separate differential equations with coefficients $\alpha(\cdot,T), \sigma(\cdot,T)$ and initial condition $f_0(T)$ for each T. Our aim is to extend (I.2.1) by replacing W by an $\mathbb{R}^d$-valued Lévy process $Z = (Z_1, \ldots, Z_d)$. Then (I.2.1) boils down to

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sum_{i=1}^{d} \sigma_i(t,T)dZ_i(t), \qquad t \in [0,T], \quad T \geq 0, \\ f(0,T) &= f_0(T), \quad T \geq 0. \end{aligned} \tag{I.2.2}$$

The equation (I.2.2) extends the previous model framework significantly by admitting a large class of noise distributions and incorporating new path properties of forward rates, like jumps.

Factor Models

Bond prices and forward rates can also be treated as functions of *time to maturity*. For a fixed date t we focus now on the functions

$$x \mapsto P(t, t+x), \quad x \mapsto f(t, t+x), \quad x \geq 0,$$

where $x := T - t$ with $T \geq t$. Modelling the shapes of the preceding functions and their stochastic evolution in time is encompassed by the *factor models*

$$P(t,T) = F(T-t, X(t)), \quad f(t,T) = G(T-t, X(t)), \quad 0 \leq t \leq T, \tag{I.2.3}$$

where F, G are deterministic functions and X is some stochastic process bringing randomness to the model. The process X is called a *factor* and should be interpreted as consisting of observed economical parameters. In particular, it can be given by the short-rate process $R(t)$.

We study models (I.2.3) where X is a Markov process and characterize admissible functions F, G in (I.2.3), in terms of the transition semigroup of X.

Of prime interest are factors specified by stochastic equations, like the well-known Cox–Ingersol–Ross short-rate model

$$dR(t) = (aR(t) + b)dt + \sqrt{c}\sqrt{R(t)}dW(t), \quad R(0) = R_0, \quad t > 0,$$

or Vasiček, Ho–Lee and Hull–White models (see Björk [16], Filipović [52] for details). We go, however, beyond the continuous paths framework and deal also with multiplicative factors of the form

$$dX(t) = aX(t)dt + bX(t)dZ(t),\ X(0) = x,$$

as well as with the Ornstein–Uhlenbeck short-rate process

$$dR(t) = (a + bR(t))dt + dZ(t), \quad R(0) = R_0, \quad t \geq 0,$$

where Z is a Lévy process and a, b some constants.

Affine Term Structure Models
In the affine term structure model the bond prices have the form

$$P(t,T) = e^{-C(T-t)-D(T-t)R(t)}, \quad 0 \leq t \leq T, \tag{I.2.4}$$

where C, D are deterministic regular functions and R stands for the short-rate process. In fact, (I.2.4) is a particular case of (I.2.3) with

$$G(u,x) = C'(u) + D'(u)x \tag{I.2.5}$$

and the random factor given by the short-rate process R. The linear dependence over x in (I.2.5) implied by (I.2.4) allows us to characterize Lévy processes Z, which generate short rates of the form

$$dR(t) = F(R(t))dt + \sum_{i=1}^{d} G_i(R(t-))dZ_i(t), \quad t \geq 0,\ R(0) = x \tag{I.2.6}$$

that are admissible for affine models. Among real valued Lévy martingales, the only ones turn out to be the Wiener process and the α-stable martingale with Lévy measure

$$\nu(dy) = \frac{1}{y^{1+\alpha}} \mathbf{1}_{[0,+\infty)}(y)dy, \quad \alpha \in (1,2).$$

In the multidimensional case the coordinates of Z can be given by the Wiener process, the α-stable martingales with $\alpha \in (1,2)$, α-stable subordinators with $\alpha \in (0,1)$ and an arbitrary subordinator that enters (I.2.6) in the additive way.

We also present a general characterization of admissible Markov short rates that generate affine models in terms of their generators. This part of the material is based on the paper [53] of Filipović and also provides a characterization of the functions C, D in (I.2.4).

Constructing Models
An efficient way to construct arbitrage-free models is by using the theory of partial differential equations for forward rate processes. The no-arbitrage requirement leads

to equations with nonlocal and nonlinear coefficients. A typical example is the following equation for the forward rate

$$r(t,x) = \left(\frac{\partial}{\partial x} r(t,x) + \left(\int_0^x r(t,v)dv \right)^{\alpha} r(t,x) \right) dt + r(t,x)dZ(t), \quad x \geq 0, t \geq 0,$$

where

$$r(t,x) := f(t,t+x), \quad x \geq 0, t \geq 0,$$

and Z is an α-stable martingale. Equations of this type with the Wiener process Z were introduced by Musiela [97]. The equations prompt interesting mathematical questions about existence and uniqueness of solutions and their positivity, discussed in Part IV.

I.3 Content of the Book

The book consists of four parts: (I) “Bond Market in Discrete Time”; (II) “Fundamentals of Stochastic Analysis”; (III) “Bond Market in Continuous Time”; and (IV) “Stochastic Equations in the Bond Market”. The first part has a more elementary character than the remaining three. It uses classical probability concepts and results rather than more advanced stochastic analysis, as in the rest of the book. The book ends with Appendices containing the proof of the martingale representation theorem in the pure jump case, material on generators of equations with Lévy processes and on evolution equations. Special care is devoted to the following models of the bond market: the HJM model in which forward rates are defined by stochastic equations; factor models in which price curves are moved by stochastic processes of economic factors; and affine models in which bond prices are exponential functions of the short rate.

Part I starts from preliminaries on the discrete time financial market in Chapter 1. Arbitrage-free models are studied in Chapter 2. We derive, in particular, a discrete time version of the CIR equation of the continuous time theory. Practically, all Markovian short-rate processes of the affine term structure are determined. Completeness of the bond market is studied in Chapter 3. Bond curves are vectors with an infinite number of coordinates and only those models with curves evolving in finite dimensional spaces might be complete. Specific conditions for approximate completeness of the main models are deduced.

Part II is divided into three substantial chapters. Chapter 4 recalls concepts and results from stochastic analysis like semimartingales, square integrable martingales and Doob–Meyer decomposition. Stochastic integration with respect to semimartingales and random measures as well as Ito’s formula are treated. They will be of constant use later. Chapter 5 concerns Lévy processes, our basic tool. We first apply the general stochastic analysis theory to this class of processes and describe specific subclasses. Then in Chapter 6 we formulate the integral representation theorem for local martingales with respect to the Lévy filtration due to Kunita. Essential, although rather classical elements of the proof, like chaos expansion and multiple Itô–Wiener integrals are presented in Appendix A. The second part of this chapter is concerned with Girsanov's formula for densities of equivalent measures.

Part III concerning the continuous time bond market starts with a mathematical description of the models and their elementary properties in Chapter 7. Arbitrage-free Heath–Jarrow–Morton models of the bond market are analyzed in Chapter 8. The main results here are general non-arbitrage conditions of the HJM type for an arbitrary physical probability measure. Chapter 9 investigates the non-arbitrage problem when the models are given in the form of forward curves moved by Markovian factor processes. The main result is the *term structure equation*. Some applications to special factor processes, like multiplicative or Ornstein–Uhlenbeck processes are presented as well. Chapter 10 is devoted to non-arbitrage conditions for affine models of bond prices. It consists of two major sections concerned, respectively, with short rates given as solutions of general stochastic equations and short rates that are general Markov processes. We present results due to Filipović. We also give a generalization to discontinuous short-rate processes of the Cox–Ingersol–Ross theorem. The final Chapter 11 is on completeness of the bond market. The hedging problem is formulated in terms of the solvability of the so-called *hedging equation*. It is discussed in various settings related to special forms of the Lévy process. Approximate completeness is discussed as well.

Part IV focuses on building arbitrage-free markets through stochastic equations. In Chapter 12 the equations are introduced. General equations for the forward curve under the martingale measure are analyzed by the methods of stochastic evolution equations in Chapter 13. Conditions for local and global existence of solutions are established. Some applications to the so-called Morton–Musiela equation are presented as well. Chapter 14 treats the case when volatility in the HJM model is a linear function of the forward curve. Then the forward rate satisfies the so-called Morton's equation. The equation has a unique solution for a large class of Lévy processes characterized in terms of the logarithmic growth conditions of the Lévy exponent. We develop the method introduced by Morton, who treated the Wiener case and obtained a negative result. The Morton–Musiela equation is treated in Chapter 15.