

Contents

<i>Preface</i>	<i>page xi</i>
1 Linear Volterra Integral Equations	1
1.1 Introduction	1
1.2 Second-Kind VIEs with Smooth Kernels	2
1.2.1 Existence and Uniqueness of Solutions	2
1.2.2 Linear VIEs with Convolution Kernels	11
1.2.3 Adjoint VIEs	13
1.2.4 Systems of Linear VIEs	14
1.2.5 Comparison Theorems	17
1.3 Second-Kind VIEs with Weakly Singular Kernels	18
1.3.1 The Mittag-Leffler Function and Weakly Singular VIEs	19
1.3.2 Existence and Uniqueness of Solutions	21
1.3.3 Other Types of Singular Kernels	26
1.3.4 Comparison Theorems	27
1.4 VIEs of the First Kind: Smooth Kernels	29
1.4.1 Existence and Uniqueness of Solutions	29
1.4.2 First-Kind VIEs are Ill-Posed Problems	33
1.5 VIEs of the First Kind with Weakly Singular Kernels	36
1.5.1 Abel's Integral Equation	36
1.5.2 General First-Kind VIEs: Volterra's Nota II of 1896	39
1.5.3 Other Types of Kernel Singularities	42
1.6 VIEs of the Third Kind (I)	45
1.7 Exercises and Research Problems	48
1.8 Notes	51
2 Regularity of Solutions	57
2.1 VIEs of the Second Kind	57

2.1.1	VIEs with Smooth Kernels	57
2.1.2	VIEs with Weakly Singular Kernels	58
2.1.3	Bounded but Non-Smooth Kernels	63
2.1.4	Kernels with Boundary Singularities	64
2.1.5	Kernel Singularities of the Form $(t^2 - s^2)^{-1/2}$	65
2.2	VIEs of the First Kind	67
2.2.1	VIEs with Smooth Kernels	67
2.2.2	VIEs with Weakly Singular Kernels	68
2.2.3	Other Types of Kernel Singularities	70
2.2.4	The Generalised Abel Integral Equation	71
2.3	Linear Volterra Functional Integral Equations	73
2.3.1	Introduction	73
2.3.2	Second-Kind VFIEs with Vanishing Delays	76
2.3.3	First-Kind VFIEs with Vanishing Delays	81
2.3.4	Second-Kind VFIEs with Non-Vanishing Delays	84
2.3.5	First-Kind VFIEs with Non-Vanishing Delays	90
2.4	Exercises and Research Problems	95
2.5	Notes	100
3	Non-Linear Volterra Integral Equations	103
3.1	Non-Linear Second-Kind VIEs	103
3.1.1	General Existence Theorems	103
3.1.2	VIEs of Hammerstein Type	111
3.1.3	Maximal Solutions and a Comparison Theorem	112
3.1.4	VIEs with Multiple Solutions	116
3.1.5	Regularity Results	119
3.2	Solutions with Finite-Time Blow-Up	121
3.2.1	Introduction	121
3.2.2	Blow-Up Theory for General Hammerstein VIEs	127
3.2.3	Multiple Solutions – Revisited	132
3.3	Quenching of Solutions	134
3.3.1	Quenching in Differential Equations	134
3.3.2	Quenching in VIEs of Hammerstein Type	138
3.4	Other Types of Non-Linear VIEs	144
3.4.1	Non-Standard Second-Kind VIEs	145
3.4.2	VIEs of Auto-Convolution Type	146
3.4.3	Implicit VIEs	153
3.5	Non-Linear First-Kind VIEs	157
3.6	Non-Linear Volterra Functional Integral Equations	159
3.6.1	State-Independent Delays	159

Contents

vii

3.6.2	Blow-Up Theory for Non-Linear VFIEs	161
3.6.3	State-Dependent Delays	163
3.7	Exercises and Research Problems	164
3.8	Notes	169
4	Volterra Integral Equations with Highly Oscillatory Kernels	175
4.1	Introduction	175
4.2	VIEs of the Second Kind	176
4.2.1	Smooth Kernels	176
4.2.2	VIEs with Weakly Singular Kernels	179
4.2.3	Comparison with Highly Oscillatory Fredholm Integral Equations	183
4.3	VIEs of the First Kind	184
4.3.1	VIEs with Smooth Kernels	184
4.3.2	VIEs with Weakly Singular Kernels	189
4.3.3	Other Types of Oscillators	192
4.4	General Oscillators $e^{i\omega g(t,s)}$	193
4.5	Exercises and Research Problems	193
4.6	Notes	196
5	Singularly Perturbed and Integral-Algebraic Volterra Equations	198
5.1	Singularly Perturbed VIEs	198
5.1.1	Examples	199
5.1.2	VIEs with Smooth Kernels	202
5.1.3	VIEs with Weakly Singular Kernels	205
5.1.4	Non-Linear VIEs	207
5.2	Integral-Algebraic Equations with Smooth Kernels	207
5.2.1	Introduction	207
5.2.2	ν -smoothing Volterra Integral Operators	210
5.2.3	The Tractability Index of a System of Linear IAEs	211
5.2.4	The Decoupling of Index-1 IAEs	215
5.3	Open Problems	217
5.4	Exercises and Research Problems	218
5.5	Notes	220
6	Qualitative Theory of Volterra Integral Equations	223
6.1	Introduction	223
6.2	Asymptotic Properties of Resolvent Kernels	224
6.2.1	VIEs with Convolution Kernels	224
6.2.2	VIEs with General Kernels	229

6.3	Asymptotic Behaviour of Solutions	229
6.3.1	VIEs with Convolution Kernels	229
6.4	VIEs of Hammerstein Form	231
6.4.1	Non-Linear Perturbations of Linear VIEs	231
6.4.2	Hammerstein VIEs with Convolution Kernels	233
6.5	Exercises and Research Problems	236
6.6	Notes	238
7	Cordial Volterra Integral Equations	241
7.1	Cordial Volterra Integral Operators	241
7.1.1	Basic Properties of Cordial Volterra Integral Operators	242
7.1.2	The Spectrum of a Cordial Volterra Integral Operator	249
7.2	Linear Cordial Volterra Integral Equations	252
7.2.1	Cordial VIEs of the Second Kind	252
7.2.2	Cordial VIEs of the First Kind	255
7.2.3	VIEs of the Third Kind (II)	261
7.3	Non-Linear Cordial VIEs	266
7.4	Cordial VIEs with Highly Oscillatory Kernels	270
7.4.1	The Spectra of Highly Oscillatory Cordial Volterra Operators	270
7.4.2	Second-Kind Cordial VIEs with Highly Oscillatory Kernels	272
7.4.3	Cordial First-Kind VIEs with Highly Oscillatory Kernels	274
7.5	Exercises and Research Problems	274
7.6	Notes	277
8	Volterra Integral Operators on Banach Spaces	279
8.1	Mapping Properties	279
8.1.1	Volterra Integral Operators on $C(I)$	279
8.1.2	Volterra Integral Operators on Hölder and L^p -spaces	280
8.2	Quasi-Nilpotency	284
8.3	Resolvent Kernels and Resolvent Operators	285
8.3.1	Resolvent Kernels for Second-Kind VIEs	285
8.3.2	Resolvent Kernels for First-Kind VIEs	286
8.4	Singular Values of Volterra Integral Operators	288
8.5	Norms of Powers of \mathcal{V}	292
8.5.1	The Basic Volterra Integral Operator \mathcal{V}	292
8.5.2	Volterra Integral Operators with Convolution Kernels	297
8.6	Exercises and Research Problems	297
8.7	Notes	300

9	Applications of Volterra Integral Equations	304
9.1	VIEs of the First Kind	304
9.1.1	Integral Equations of Abel Type	304
9.1.2	General First-Kind VIEs	305
9.2	VIEs of the Second Kind	307
9.2.1	The Renewal Equation	307
9.2.2	Population Growth Models	308
9.2.3	Heat Transfer, Diffusion Models and Shock Wave Problems	309
9.2.4	Blow-Up and Quenching Phenomena	312
9.2.5	American Option Pricing	315
9.2.6	Optimal Control Problems	316
9.2.7	A Brief Review of Further Applications	317
9.3	VIEs of the Third Kind	320
9.4	Systems of Integral-Algebraic VIEs	320
9.5	Notes	322
Appendix	A Review of Banach Space Tools	325
A.1	Banach Spaces in the Theory of VIEs	325
A.1.1	The Spaces $C^d(I)$	325
A.1.2	The Hölder Spaces $C^{d,\beta}(I)$	326
A.1.3	The Lebesgue Spaces $L^p(0, T)$	327
A.1.4	The Sobolev Spaces $W^{d,p}(\Omega)$	329
A.2	Linear Operators on Banach Spaces	330
A.2.1	Bounded Operators	330
A.2.2	Compact Operators	332
A.2.3	The Spectrum of Bounded Linear Operators	335
A.2.4	Quasi-Nilpotent Operators	338
A.3	Non-Linear Operators on Banach Spaces	339
A.3.1	The Fréchet Derivative	339
A.3.2	The Implicit Function Theorem	340
A.3.3	The Fixed-Point Theorems of Banach and Schauder	341
A.4	Notes	342
	<i>References</i>	344
	<i>Index</i>	383