# Contents

**Preface**  
page xi

1 **Linear Volterra Integral Equations**  
1.1 Introduction  
1.2 Second-Kind VIEs with Smooth Kernels  
1.2.1 Existence and Uniqueness of Solutions  
1.2.2 Linear VIEs with Convolution Kernels  
1.2.3 Adjoint VIEs  
1.2.4 Systems of Linear VIEs  
1.2.5 Comparison Theorems  
1.3 Second-Kind VIEs with Weakly Singular Kernels  
1.3.1 The Mittag-Leffler Function and Weakly Singular VIEs  
1.3.2 Existence and Uniqueness of Solutions  
1.3.3 Other Types of Singular Kernels  
1.3.4 Comparison Theorems  
1.4 VIEs of the First Kind: Smooth Kernels  
1.4.1 Existence and Uniqueness of Solutions  
1.4.2 First-Kind VIEs are Ill-Posed Problems  
1.5 VIEs of the First Kind with Weakly Singular Kernels  
1.5.1 Abel’s Integral Equation  
1.5.2 General First-Kind VIEs: Volterra’s Nota II of 1896  
1.5.3 Other Types of Kernel Singularities  
1.6 VIEs of the Third Kind (I)  
1.7 Exercises and Research Problems  
1.8 Notes  

2 **Regularity of Solutions**  
2.1 VIEs of the Second Kind  

© in this web service Cambridge University Press  
www.cambridge.org
## Contents

2.1.1 VIEs with Smooth Kernels 57  
2.1.2 VIEs with Weakly Singular Kernels 58  
2.1.3 Bounded but Non-Smooth Kernels 63  
2.1.4 Kernels with Boundary Singularities 64  
2.1.5 Kernel Singularities of the Form \((t^2 - s^2)^{-1/2}\) 65  
2.2 VIEs of the First Kind 67  
  2.2.1 VIEs with Smooth Kernels 67  
  2.2.2 VIEs with Weakly Singular Kernels 68  
  2.2.3 Other Types of Kernel Singularities 70  
  2.2.4 The Generalised Abel Integral Equation 71  
2.3 Linear Volterra Functional Integral Equations 73  
  2.3.1 Introduction 73  
  2.3.2 Second-Kind VFIEs with Vanishing Delays 76  
  2.3.3 First-Kind VFIEs with Vanishing Delays 81  
  2.3.4 Second-Kind VFIEs with Non-Vanishing Delays 84  
  2.3.5 First-Kind VFIEs with Non-Vanishing Delays 90  
2.4 Exercises and Research Problems 95  
2.5 Notes 100  
3 Non-Linear Volterra Integral Equations 103  
  3.1 Non-Linear Second-Kind VIEs 103  
    3.1.1 General Existence Theorems 103  
    3.1.2 VIEs of Hammerstein Type 111  
    3.1.3 Maximal Solutions and a Comparison Theorem 112  
    3.1.4 VIEs with Multiple Solutions 116  
    3.1.5 Regularity Results 119  
  3.2 Solutions with Finite-Time Blow-Up 121  
    3.2.1 Introduction 121  
    3.2.2 Blow-Up Theory for General Hammerstein VIEs 127  
    3.2.3 Multiple Solutions – Revisited 132  
  3.3 Quenching of Solutions 134  
    3.3.1 Quenching in Differential Equations 134  
    3.3.2 Quenching in VIEs of Hammerstein Type 138  
  3.4 Other Types of Non-Linear VIEs 144  
    3.4.1 Non-Standard Second-Kind VIEs 145  
    3.4.2 VIEs of Auto-Convolution Type 146  
    3.4.3 Implicit VIEs 153  
  3.5 Non-Linear First-Kind VIEs 157  
  3.6 Non-Linear Volterra Functional Integral Equations 159  
    3.6.1 State-Independent Delays 159
## Contents

3.6.2 Blow-Up Theory for Non-Linear VFIEs 161
3.6.3 State-Dependent Delays 163
3.7 Exercises and Research Problems 164
3.8 Notes 169

4 Volterra Integral Equations with Highly Oscillatory Kernels 175
4.1 Introduction 175
4.2 VIEs of the Second Kind 176
4.2.1 Smooth Kernels 176
4.2.2 VIEs with Weakly Singular Kernels 179
4.2.3 Comparison with Highly Oscillatory Fredholm Integral Equations 183
4.3 VIEs of the First Kind 184
4.3.1 VIEs with Smooth Kernels 184
4.3.2 VIEs with Weakly Singular Kernels 189
4.3.3 Other Types of Oscillators 192
4.4 General Oscillators $e^{ig(t,s)}$ 193
4.5 Exercises and Research Problems 193
4.6 Notes 196

5 Singularly Perturbed and Integral-Algebraic Volterra Equations 198
5.1 Singularly Perturbed VIEs 198
5.1.1 Examples 199
5.1.2 VIEs with Smooth Kernels 202
5.1.3 VIEs with Weakly Singular Kernels 205
5.1.4 Non-Linear VIEs 207
5.2 Integral-Algebraic Equations with Smooth Kernels 207
5.2.1 Introduction 207
5.2.2 $\nu$-smoothing Volterra Integral Operators 210
5.2.3 The Tractability Index of a System of Linear IAEs 211
5.2.4 The Decoupling of Index-1 IAEs 215
5.3 Open Problems 217
5.4 Exercises and Research Problems 218
5.5 Notes 220

6 Qualitative Theory of Volterra Integral Equations 223
6.1 Introduction 223
6.2 Asymptotic Properties of Resolvent Kernels 224
6.2.1 VIEs with Convolution Kernels 224
6.2.2 VIEs with General Kernels 229
### Contents

6.3 Asymptotic Behaviour of Solutions .................................................. 229
  6.3.1 VIEs with Convolution Kernels ............................................... 229

6.4 VIEs of Hammerstein Form ............................................................. 231
  6.4.1 Non-Linear Perturbations of Linear VIEs ..................................... 231
  6.4.2 Hammerstein VIEs with Convolution Kernels .................................. 233

6.5 Exercises and Research Problems ..................................................... 236

6.6 Notes ................................................................................................. 238

7 Cordial Volterra Integral Equations ....................................................... 241
  7.1 Cordial Volterra Integral Operators .................................................. 241
    7.1.1 Basic Properties of Cordial Volterra Integral Operators .................. 242
    7.1.2 The Spectrum of a Cordial Volterra Integral Operator .................... 249
  7.2 Linear Cordial Volterra Integral Equations ........................................ 252
    7.2.1 Cordial VIEs of the Second Kind .............................................. 252
    7.2.2 Cordial VIEs of the First Kind ............................................... 255
    7.2.3 VIEs of the Third Kind (II) ...................................................... 261
  7.3 Non-Linear Cordial VIEs ................................................................. 266
  7.4 Cordial VIEs with Highly Oscillatory Kernels .................................... 270
    7.4.1 The Spectra of Highly Oscillatory Cordial Volterra Operators .......... 270
    7.4.2 Second-Kind Cordial VIEs with Highly Oscillatory Kernels ............... 272
    7.4.3 Cordial First-Kind VIEs with Highly Oscillatory Kernels .................. 274
  7.5 Exercises and Research Problems .................................................... 274

7.6 Notes ................................................................................................. 277

8 Volterra Integral Operators on Banach Spaces ....................................... 279
  8.1 Mapping Properties ........................................................................... 279
    8.1.1 Volterra Integral Operators on \( C(I) \) ........................................... 279
    8.1.2 Volterra Integral Operators on Hölder and \( L^p \)-spaces .................. 280
  8.2 Quasi-Nilpotency .............................................................................. 284

8.3 Resolvent Kernels and Resolvent Operators ....................................... 285
  8.3.1 Resolvent Kernels for Second-Kind VIEs ....................................... 285
  8.3.2 Resolvent Kernels for First-Kind VIEs ........................................... 286

8.4 Singular Values of Volterra Integral Operators .................................... 288

8.5 Norms of Powers of \( V \) ................................................................. 292
  8.5.1 The Basic Volterra Integral Operator \( V \) ......................................... 292
  8.5.2 Volterra Integral Operators with Convolution Kernels ..................... 297

8.6 Exercises and Research Problems ...................................................... 297

8.7 Notes ................................................................................................. 300
Contents

9 Applications of Volterra Integral Equations 304
  9.1 VIEs of the First Kind 304
    9.1.1 Integral Equations of Abel Type 304
    9.1.2 General First-Kind VIEs 305
  9.2 VIEs of the Second Kind 307
    9.2.1 The Renewal Equation 307
    9.2.2 Population Growth Models 308
    9.2.3 Heat Transfer, Diffusion Models and Shock Wave Problems 309
    9.2.4 Blow-Up and Quenching Phenomena 312
    9.2.5 American Option Pricing 315
    9.2.6 Optimal Control Problems 316
    9.2.7 A Brief Review of Further Applications 317
  9.3 VIEs of the Third Kind 320
  9.4 Systems of Integral-Algebraic VIEs 320
  9.5 Notes 322

Appendix A Review of Banach Space Tools 325
  A.1 Banach Spaces in the Theory of VIEs 325
    A.1.1 The Spaces $C^d (I)$ 325
    A.1.2 The Hölder Spaces $C^{d,\beta} (I)$ 326
    A.1.3 The Lebesgue Spaces $L^p (0, T)$ 327
    A.1.4 The Sobolev Spaces $W^{d,p} (\Omega)$ 329
  A.2 Linear Operators on Banach Spaces 330
    A.2.1 Bounded Operators 330
    A.2.2 Compact Operators 332
    A.2.3 The Spectrum of Bounded Linear Operators 335
    A.2.4 Quasi-Nilpotent Operators 338
  A.3 Non-Linear Operators on Banach Spaces 339
    A.3.1 The Fréchet Derivative 339
    A.3.2 The Implicit Function Theorem 340
    A.3.3 The Fixed-Point Theorems of Banach and Schauder 341
  A.4 Notes 342

References 344
Index 383