Index

$BC_0(\mathbb{R}^+)$, 236
$C^d(I)$, 325
$C^{d,p}(I)$, 326
inclusion relation, 327
$L^p(0,T)$, 327
inclusion relation, 328
$L^1_{loc}(\mathbb{R}^+)$, 227
$W^{d,p}(0,T)$, 329
inclusion relation, 329
$K(X,Y)$, 332
$L(X,Y)$, 330
$V$, 2
$V_{d}$, 18
$V_{o,a}$, 175
$V_{o,0}$, 76
$V_{e}$, 242
$V_{e,1}$, 243
$V_{e,0}$, 76

Abel integral equation, 72
adjoint, 50, 51, 72
application, 305
existence of solution, 38
generalised, 71, 97
ill posed, 97
inversion formula, 36, 43
non-standard equation, 44
resolvent kernel, 36
variation-of-constants formula, 36
Abel–Volterra integral equation, 19
abstract VIEs, 53, 302
adjoint Volterra operator, 13
auto-convolution operator, 299

auto-convolution VIE
adjoint, 151, 173
basic, 146
existence of solution, 146, 147
first kind, 147, 173
generalised, 147
regularity of solution, 151
second kind, 172
third kind, 172

Banach algebra, 335
non-commutative, 335
non-unital, 336
point spectrum, 336
resolvent set, 336
spectrum, 336
unital, 335, 336
Banach fixed-point theorem, 341
blow-up solution, 121
blow-up time, 123
breaking points, 75

Carathéodory iteration, 110, 168
Cauchy–Schwarz inequality, 328
closed graph theorem, 331
compact mapping, 341
compact operator, 332
spectrum, 337
comparison theorem
linear VIE, 17
non-linear VIE, 18, 115
weakly singular VIE, 27
completely monotone kernel, 226
completely monotone solution, 21
contraction map, 341
Index

cordial VIE
  first kind, 255
  Hammerstein form, 266
  highly oscillatory kernel, 272, 274
  non-linear, 266, 267, 270
  regularity of solution, 253, 270
  second kind, 252
  variation-of-constants, 257

cordial Volterra integral operator, 27, 242
  basic, 243
  basic properties, 242
  composition, 247, 274
  core, 242
  highly oscillatory kernel, 271
  mapping properties, 283
  non-compact, 245
  non-linear, 267
  resolvent set, 252
  spectrum, 250, 251, 271, 272

damping factor, 177, 179

decoupled system of IAEs, 208

delay
  constant, 74
  non-vanishing, 74, 84
  vanishing, 74

delay function, 73, 74, 84
  highly oscillatory, 98
  non-linear, 79
  vanishing, 81

Dirichlet’s formula, 5
Du Bois-Reymond, 51

Euler beta function, 22
Euler constant, 41

Fox–Li oscillator, 183
Fréchet derivative, 340
fractional calculus, 54
Fredholm alternative, 25, 334
Fredholm integral equation
  first kind, 72
  highly oscillatory kernel, 183, 196
  second kind, 7
  third kind, 55
Fredholm integral operator, 7
Fredholm integral equation
  second kind, 52

generalised Abel equation, 71
Gronwall lemma, 6, 17

for cordial VIE, 254

generalised, 24, 28
Hölder constant, 326
Hölder continuity, 326
Hölder inequality, 328
Hölder space, 59, 62, 72, 120, 327
Hammerstein (1930), 169
Hammerstein integral equation
  general theory, 112
Hammerstein integral operator, 104
highly oscillatory VIE, 175
asymptotic expansion of solution, 177, 181
  188
  damping of solution, 177, 186
  resolvent kernel, 176
  weakly singular kernel, 180, 189

Hilbert–Schmidt integral operator, 280

IAE
ν-smoothing, 210
consistency relation, 208
decoupled system, 215
fully decoupled system, 208
Hammerstein form, 218
index-1 system, 215
matrix chain, 212
non-linear, 218
projectors, 212
semi-explicit form, 209
smooth kernel, 207
tractability index, 212, 213
weakly singular kernel, 218
ill-posed problem, 33, 97, 334
ill-posed VIE, 33
implicit VIE, 31, 153
regularity, 154
regularity of solution, 168
implicitly linear integral equation, 112
integral inequalities, 53
integral-algebraic equation: see IAE, 207
iterated kernels, 3, 11
  for weakly singular kernel, 23

kernel
completely monotone, 226
highly oscillatory, 175
see also nullspace, 330

lag function, 73
Lalesco, 3, 84
Laplace transform, 146
Le Roux (1895), 53
Index

Lighthill’s equation, 267, 311
Lighthill’s Volterra integral operator, 251
maximal solution, 170
minimal solution, 170
Mittag-Leffler function, 19, 28, 54
completely monotone, 20
Nemytzki (1934), 170
Nemytzki operator, 111
Neumann series, 4, 292
convolution kernel, 12
system of VIEs, 15
nilpotency, 299
non-linear second-kind VIE
existence of solution, 105
uniqueness of solution, 105
non-linear VFIE
state-dependent delay, 163
non-linear VIE
auto-convolution equation, 146
continuation of solution, 170
first kind, 157
multiple solution, 116
non-standard form, 145
non-trivial solution, 117
Picard iteration, 106
standard form, 103
trivial solution, 117
weakly singular kernel, 107
non-vanishing delay, 74, 84
norm of operator, 330
null space, 330
ODE
quenching of solution, 166
open mapping theorem, 331
operator
adjoint, 338
bounded, 330
closed, 330
compact, 332, 333
graph, 330
inverse, 331
nilpotent, 284
norm, 330
projection, 332
quasi-nilpotent, 284
self-adjoint, 289, 338
substitution, 111
orthogonal projection, 332
oscillator

Fox–Li, 183
other types, 192
separable, 179
Paley–Wiener theorem, 225
partial VIEs
semi-discretisation, 14
PDE
quenching of solution, 137
quenching time, 135
Picard iteration, 3, 11, 53, 78
auto-convolution VIE, 148
non-linear VIE, 104
VFIE, 82
weakly singular VIE, 23
point spectrum, 337
population growth VFIE
state-dependent delay, 309
primary discontinuity points, 75, 85
projection operator, 332
proportional delay, 159
non-vanishing, 75
VFIE, 76
quasi-nilpotent operator, 339
quenching
differential equation, 134
VIE, 139
quenching time, 134, 139
regularity of solution
cordial VIE, 254
second-kind VIE, 58
weakly singular VFIE, 90
weakly singular VIE, 58
renewal equation, 11, 13
vector form, 16
resolvent equation, 5, 88
abstract, 286
complementary, 5, 286
convolution kernel, 12
resolvent kernel, 4, 88
abstract, 286
first kind, 287
for non-smooth kernel, 64
for weakly singular kernel, 22
integrability, 225
resolvent theory, 301
Schauder fixed point theorem, 342
Schauder-Leray fixed point theorem, 342
### Index

- **Shea–Wainger theorem**, 227  
- singular value decomposition, 290  
- singular values, 289  
- singularly perturbed VIE, 198  
- boundary layer, 200  
- Hammerstein equation, 207  
- highly oscillatory kernel, 184  
- inner solution, 200  
- outer solution, 200  
- reduced equation, 199  
- weakly singular kernel, 206  
- Sobolev space, 329  
- spectral radius, 339  
- spectral theorem, 338  
- state-dependent delay, 309  
- substitution operator, 111  
- superposition principle, 61  
- system of VIEs  
  - representation of solution, 15  
  - resolvent equations, 14  
  - resolvent kernel, 14  
  - viscoelastic problems, 14  
- time transformation, 169  
- tractability index  
  - definition, 213  
- Tricomi’s equation, 72  
- Uryson integral operator, 103  
- variation-of-constants formula, 232  
  - linear VIE, 10  
- VFIE, 87  
- VFIE of the first kind  
  - boundedness condition, 92  
  - existence of solution, 92  
  - non-vanishing delay, 90  
  - smoothing of solution, 92  
  - unbounded solution, 92  
  - vanishing delay, 82, 99  
- VHIE, 111  
- VIDE  
  - variation-of-constant formula, 8  
- VIDE: Volterra integro-differential equation, 8  
- VIE  
  - Abel–Volterra, 19  
  - abstract equations, 53, 302  
  - adjoint equation, 13, 48  
  - blow-up of solution, 108  
  - blow-up profile, 132  
  - blow-up theory, 127  
  - blow-up time, 109, 121, 123, 126, 165  
  - boundary singularities, 64  
  - comparison theorem, 17, 28  
  - convolution kernel, 11, 30  
  - cordial, 252  
  - diagonal singularity, 58  
  - finite-time blow-up, 108, 121  
  - first kind, 2, 3, 29, 54, 67  
  - Hammerstein form, 111, 233  
  - highly oscillatory kernel, 175  
  - identification of memory kernel, 317  
  - ill-posed, 33  
  - implicit, 31, 153, 173  
  - kernel singularities, 18  
  - linear oscillator, 176  
  - maximal solution, 113  
  - minimal solution, 113  
  - multiple solution, 132  
  - non-classical, 54  
  - non-linearly perturbed, 231  
  - non-standard, 145  
  - non-standard forms, 104  
  - optimal control problem, 316  
  - optimal control problems, 316  
  - quadratic, 146  
  - quenching of solution, 109, 135, 139  
  - quenching theory, 138  
  - quenching time, 110, 137  
  - regularity of solution, 58, 60, 61, 63, 64, 119  
  - renewal equation, 13  
  - representation of solution, 5  
  - second kind, 2, 3, 18
Index 387

separable oscillator, 176
singular kernel, 18, 26, 58, 65
singularly perturbed, 198
state-dependent delay, 163
system, 14
third kind, 2, 31, 46, 51, 55, 56, 261, 320
two variable limits of integration, 15
variation of constants, 10
Volterra–Stieltjes, 300, 302, 303
Volterra–Stieltjes equations, 52
weakly singular kernel, 19
VIE of the first kind
\(\nu\)-smoothing problem, 34
adjoint, 97
cordial, 255
degree of ill-posedness, 34
Hammerstein form, 30
highly oscillatory kernel, 184
logarithmic kernel, 41
mildly ill-posed, 290
regularity of solution, 40, 44, 67, 68, 70
resolvent kernel, 256, 286
resolvent representation, 256
severely ill-posed, 290
singular kernel, 36, 42, 68, 70
system, 35, 50, 97
variation-of-constants, 256
VIE of the second kind
non-linear, 103
VIE of the third kind, 45, 46
connection with cordial VIE, 261
non-linear, 268, 270
VIE with weakly singular kernel
\(\nu\)-smoothing, 42
Volterra
biography, 53
Nota I, 1, 3–5, 29, 52
Nota II, 1, 39, 50, 52, 55, 99
Nota III, 2, 46
Nota IV, 2
Volterra functional integral equation: see
VFIE, 73
Volterra inequality, 17
Volterra integral equation: see VIE, 2
Volterra integral operator
\(V\), 2
\(V_\nu\), 18
\(V_\nu^1\), 243
\(\nu\)-smoothing, 34
adjoint, 13, 43, 276, 289, 294
boundary singularity, 244
compact, 279
cordial, 242
delay argument, 76
eigenfunction, 45
eigenvalue, 45
finite rank, 11
Hammerstein, 111
linear, 2
mapping properties, 282
non-compact, 27, 45, 242
norm of powers, 292
quasi-nilpotency, 284
singular value, 288
spectrum, 7
two variable limits, 298
Volterra–Abel integral operator
non-standard, 42
Volterra–Hammerstein integral equation, 104
Volterra–Hammerstein integral operator, 111
weakly singular kernel
\(\nu\)-smoothing, 42
weakly singular VFIE
regularity of solution, 90
weakly singular VIE
representation of solution, 22, 59
well-posed problem, 33
Young inequality, 328