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Volterra Integral Equations

An Introduction to Theory and Applications

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Preface

This monograph presents an introduction to the theory of linear and non-linear Volterra integral equations, ranging from Volterra's fundamental contributions and the resulting classical theory to more recent developments. The latter include Volterra functional integral equations with various kinds of delays, Volterra integral equations with highly oscillatory kernels, and Volterra integral equations with non-compact operators. One of the aims of the book is to introduce the reader to the current state of the art in the theory of Volterra integral equations. In addition, it illustrates – by means of a representative selection of examples – the increasingly important role Volterra integral equations play in the mathematical modelling of phenomena where memory effects play a key role.

The book is intended also as a 'stepping stone' to the literature on the advanced theory of Volterra integral equations, as presented, for example, in the monumental and seminal monograph by Gripenberg (1990). The notes at the end of each chapter and the annotated references point the reader to such papers and books.

We give a brief outline of the contents of the various chapters. As will be seen, Chapters 1, 2, 3 and 6 describe what might be called the classical theory of linear and non-linear Volterra integral equations, while the other chapters are concerned with more recent developments. Those chapters will also reveal that the theory of Volterra integral equations is by no means complete, and that many challenging problems (many of which are stated as Research Problems in the Exercise sections at the ends of the chapters) remain to be addressed.

Chapter 1 contains an introduction to the basic theory of the existence and uniqueness of solutions of linear Volterra integral equations of the first and second kind, including equations with various types of integrable kernel singularities. It also gives a brief introduction to the ill-posed nature of first-kind equations (which will play a key role in Chapter 5 when analysing systems of integral-algebraic equations).

The focus of *Chapter 2* is on the regularity properties of the solutions of the linear Volterra integral equations discussed in Chapter 1. It also contains an introduction to linear functional Volterra integral equations with vanishing or non-vanishing delays: here, the regularity of the solution depends strongly on the type of delay.

Chapter 3 is concerned with the classical theory of non-linear Volterra integral and functional integral equations. In addition, we study non-linear equations whose solutions blow up in finite time (that is, the solution becomes unbounded at some finite time), or which quench in finite time (meaning that the solution remains bounded but its first derivative blows up at some finite time). We also look at some classes of 'non-standard' Volterra integral equations, including auto-convolution equations of the second kind and implicit Volterra integral equations. The chapter concludes with a short discussion of non-linear Volterra functional integral equations: we show that in the analysis of such problems with state-dependent delays, many problems are still open.

In *Chapter 4* we turn to the relatively new subject of Volterra integral equations with highly oscillatory kernels. The focus is on the oscillation properties of the solutions of such equations of the first and second kinds, and on their asymptotic expansions as the oscillation parameter tends to infinity.

Chapter 5 begins with an introduction to the theory of singularly perturbed Volterra integral equations. It then focuses on the theory of systems of so-called integral-algebraic equations: such a system may be viewed as a non-local analogue of a system of differential-algebraic equations.

In *Chapter 6* we present a selection of results on the asymptotic behaviour of resolvent kernels and solutions of linear Volterra integral equations. For equations with convolution kernels, this theory has its origin in the celebrated theorem of Paley and Wiener (1934), while for non-convolution kernels it goes back the late 1960s and then leads into functional analysis and abstract VIEs (studied in Chapter 8).

Chapter 7 contains an introduction to the recently developed theory of so-called cordial Volterra integral equations, especially equations whose underlying integral operator is a non-compact operator (e.g. on the Banach space of continuous functions). This theory also allows for an elegant analysis of the existence of solutions for certain classes of VIEs of the third kind. The chapter concludes with a brief study of cordial Volterra integral equations with highly oscillatory kernels.

Chapter 8 is on Volterra integral operators in the setting of Banach spaces. It focuses on the mapping properties of these operators on the space of continuous functions, on Hölder spaces, and on L^p spaces. It then deals with the quasi-nilpotency and with singular values of Volterra integral operators. In addition, we present an introduction to the asymptotic behaviour of the norms of powers of Volterra integral operators.

Volterra integral equations of various kinds have been playing an increasingly important role in the mathematical modelling of physical, biological and other phenomena that are governed by memory effects. Thus, the aim of *Chapter 9* is to describe, by means of representative examples (and additional references), the wide spectrum of applications of VIEs

Finally, in the *Appendix* we collect definitions and theorems from Functional Analysis, especially from Banach space and operator theory, which are used in earlier chapters of this book, for example, in Chapter 5 (in the analysis of integral-algebraic equations), in Chapter 7 (analysis of cordial Volterra integral equations), and in Chapter 8.

Owing to limitations of space, there are various topics that could not be included in this book. Among these are Volterra–Stieltjes integral equations and stochastic Volterra integral equations (which, due to recent progress in their theory and applications, would merit a separate monograph). However, a short guide to some key references can be found at the end of the Notes to Chapter 9.

At the end of each chapter the reader will find exercises and extensive notes. The *Exercises* range from 'hands-on' problems (intended to illustrate and complement the theory of the respective chapter) to research topics of various degrees of difficulty; these will often include challenging *Research Problems*. The objectives of the *Notes* are twofold: they contain remarks complementing the contents of the given chapter (giving e.g. the sources of original results); and they serve as a guide to papers and books on the more advanced theory of VIEs, and to the literature on aspects of VIEs not treated in the book. At the end of the Notes we also point out a selection of key references on the numerical analyses of the VIEs treated in that particular chapter and identify open problems that may be of interest to the numerical analyst.

Many of the chapters should also be of interest to numerical analysts, since the derivation of a 'good' numerical method (in the sense of reflecting the essential properties of the given problem, especially the regularity of the solution) for solving Volterra integral equations depends crucially on a thorough understanding of the underlying theory. Thus, a secondary aim of this monograph (especially Chapters 2, 3, 4 and 7) is to provide this basis.

The list of *References* is intended to be representative rather than exhaustive. Also, in order to make this extensive list more useful and give it a guiding role (especially to more advanved topics not treated in his book), many of its items have been annotated, so as to enhance the Notes given at the end of each chapter: the brief comments are either cross-references to related work, or they give an idea of the main content of a paper, or point to books and survey articles containing large bibliographies complementing the information in this monograph.

In view of the stated aims of the book, and for the sake of clarity and exposition, we shall, instead of stating a particular theorem and its proof in full generality, often present a representative particular case, so as more clearly to provide insight into the key features (and the basic mathematical techniques underlying the proof of the theorem's general version), and then guide the reader to references where he/she can find the general versions and the detailed proofs. Finding a good balance between describing technical details and conveying the spirit of theorems and their proofs is also to some extent governed by the limitation of space.

This monograph is also intended for senior undergraduate and postgraduate students who wish to acquire a thorough understanding of the basic theory of Volterra integral equations and their role in mathematical modelling. Thus, the book can be used as a textbook for an upper undergraduate or a graduate course on Volterra integral equations. An undergraduate course could, for example, be based on Chapter 1 (except possibly Section 1.6), Chapter 2 (except Section 2.3), Chapter 3 (except Section 3.6), Chapter 4, Chapter 6 (e.g. Sections 6.1.1, 6.1.2; 6.2.1, 6.2.2), and selected sections of Chapter 9. A postgraduate course would likely focus on Chapters 2, 3, 5, 6, 7 and 8.

The motivation for writing this book has its origin in a series of lectures I was invited to present during a two-week postgraduate summer school at Harbin Institute of Technology (HIT) in July 2010, which was attended by some 80 PhD students and by a number of junior and established researchers from across mainland China. The numerous questions raised during the discussions pointed both to the lack of a book providing a comprehensive introduction to the basic theory of Volterra integral equations and to recent advances in this field (including applications), and hence to a lack of senior undergraduate or postgraduate lecture courses (complementing the ubiquitous courses on the theory of ordinary differential equations). I am grateful to Professors Lin Qun (Chinese Academy of Sciences), Liu Mingzhu and Song Minghui (HIT) for giving me the opportunity to present these lectures, which eventually led to this book: I hope that it will contribute to closing the above-mentioned gaps.

The many inspiring discussions with colleagues in Europe and China have allowed me to gain deeper – and often quite unexpected – new insights into various aspects of my research on the theory and numerical analysis of Volterra integral equations. In particular, I wish to express my gratitude to Professor Arieh Iserles (DAMTP, University of Cambridge), especially for inviting me to participate in the six-month programme on *Highly Oscillatory Problems* at the Isaac Newton Institute during the first half of 2007; to Professor Gennadi Vainikko (University of Tartu, Estonia) for discussions that led to the theory of cordial VIEs; and to Professor Roswitha März (Humboldt University, Berlin) for many illuminating conversations on the theory and numerical analysis of differential-algebraic equations. My interest in delay differential equations, and hence in Volterra functional integral equations, began during my visits to the University of Trieste (Italy) and my subsequent collaboration with Professors Alfredo Bellen, Lucio Torelli and, especially, Stefano Maset. I owe them many thanks, as I do my colleagues and friends in China, in particular Professors Lin Qun, Yan Ningning, Xie Hehu and Zhou Aihui (Academy of Mathematics and Systems Science, Chinese Academy of Sciences (CAS), Beijing); Professor Han Houde (Tsinghua University, Beijing); and Professor Huang Qiumei (Beijing University of Technology). It is a pleasure, too, to thank them all for the generous hospitality they extended to me during numerous visits.

I would also like to express my gratitude to my collaborators in China: Professors Zhang Ran (Jilin University, Changchun), Liang Hui (Heilongjiang University, Harbin), Dr Yang Zhan-wen (Academy of Fundamental and Interdisciplinary Science, Harbin Institute of Technology), Professor Xie Hehu (Academy of Mathematics and Systems Sciences, CAS, Beijing), and Professor Huang Qiumei (Beijing University of Technology), as well as Dr Sonia Seyed Allaei (Instituto Superior Tecnico, University of Lisbon). They all visited me for extended periods of time at Hong Kong Baptist University.

Professor Gustav Gripenberg (Aalto University, Finland), Professor Gennadi Vainikko (University of Tartu, Estonia) and Professor Martin Stynes (Beijing Computational Science Research Center) kindly agreed to read the manuscript of this book. I am deeply indebted to them for their valuable comments and suggestions. I am also pleased to thank Professor Liang Hui and Dr Yang Zhanwen for their careful checking of the manuscript.

Most of my research on topics described in this book (especially in Chapters 3, 4, 5 and 7) was carried out while I held a Visiting Professorship (2006–2011) and a Research Professorship (2011–) at Hong Kong Baptist University (HKBU). It is a pleasure to thank the Department of Mathematics and HKBU for their generous financial and technical support. I also wish gratefully

to acknowledge the hospitality and the technical support I have received from the Department of Mathematics and Statistics at the University of Strathclyde (Glasgow, Scotland) since I became a Visiting Professor in 2003, and from the Department of Mathematics and Statistics at Dalhousie University (Halifax, Nova Scotia, Canada), where I have been an Adjunct Professor since 2006. The superb collection of mathematical journals at the Killam Library of my home university, Memorial University of Newfoundland, St John's, Canada, has been an invaluable help in carrying out my research work and in writing this book.

A significant part of my research that led to this monograph has been made possible by the Natural Sciences and Engineering Research Council (NSERC) of Canada and the Hong Kong Research Grants Council (RGC) by means of numerous individual research grants. I am particularly grateful to the RGC, whose generous GRF grants made it possible to extend financial support to my research collaborators listed above during their visits to HKBU.

Finally, I would like to thank the Cambridge University Press editorial staff for their kind and invaluable support and advice: David Tranah for precious initial advice (especially on the organisation of the introductory Chapters 1 and 2), and Sam Harrison and Clare Dennison for their encouragement and for guiding the manuscript to the present printed form.

I dedicate this book to Ruth, who has opened her beautiful home *Sunnezyt* and its enchanted garden to me for many years.