1
Linear Volterra Integral Equations

Summary

This chapter presents an introduction to the history and the classical theory of linear Volterra integral equations of the first and second kinds, including equations with weakly singular kernels. The focus of the presentation is on the existence and uniqueness of solutions of such equations. More advanced properties (e.g. the regularity of solutions) of such integral equations will be studied in Chapter 2. There, we shall also present an introduction to the theory of linear Volterra functional integral equations with various types of delay arguments.

1.1 Introduction

Vito Volterra (1860–1940) presented his celebrated theory of integral equations that now bear his name in four papers in 1896. In his *Nota I* he observes the lack of a systematic means of 'inverting definite integrals', except for particular cases, and that neither does there appear to be a systematic way to determine the existence and uniqueness of a solution of this problem. By inversion of a definite integral he means the problem of finding, for given continuous (real-valued) functions $H = H(x, y)$ and $f = f(y)$, a continuous function $\varphi = \varphi(y)$ satisfying the equation

$$f(y) - f(a) = \int_a^y \varphi(x) H(x, y)\, dx$$

on a given interval $[a, a + A]$, assuming that $H(y, y) \neq 0$ on this interval.

In *Nota II* he extends the analysis to unbounded kernels $H(x, y) = G(x, y)/(y - x)^\lambda$ with $0 < \lambda < 1$, still under the assumption that $H(y, y)$ does not vanish on $[a, a + A]$.

This assumption is dropped in *Nota III*: $H(y, y)$ is now allowed to vanish for $y = a$ but is non-zero at any other point in $(a, a + A]$. In *Nota IV* he derives sufficient conditions for the existence of a unique continuous solution when $H(y, y)$ again vanishes only for $y = a$, and f and H are of the form

$$f(y) = y^{n+1} f_1(y), \qquad H(x, y) = \sum_{i=0}^{n} a_i x^i y^{n-i} + H_1(x, y),$$

for some integer $n \geq 0$ and smooth functions f_1 and H_1.

Switching now to a notation that is commonly employed today, let $I := [0, T]$ be a given bounded and closed interval, and suppose that $f = f(t),\ g = g(t),\ a = a(t)$ and $K = K(t, s)$ are given functions with respective domains I and $D := \{(t, s) :\ 0 \leq s \leq t \leq T\}$. The equation

$$u(t) = f(t) + \int_0^t K(t, s)u(s)\, ds, \quad t \in I \tag{1.1.1}$$

is called a Volterra integral equation (VIE) of the *second kind* with *kernel* K. A Volterra integral equation of the *first kind* has the form

$$\int_0^t K(t, s)u(s)\, ds = g(t), \quad t \in I. \tag{1.1.2}$$

If the given function a vanishes at *some* points in I (but does not vanish identically on I), the integral equation

$$a(t)u(t) = f(t) + \int_0^t K(t, s)u(s)\, ds, \quad t \in I \tag{1.1.3}$$

is called a Volterra integral equation of the *third kind*. Unless stated otherwise, it will be assumed throughout this book that the given functions in (1.1.1), (1.1.2) and (1.1.3) are real-valued.

1.2 Second-Kind VIEs with Smooth Kernels

1.2.1 Existence and Uniqueness of Solutions

For a given interval $I := [0, T]$ let $\mathcal{V} :\ C(I) \to C(I)$ denote the linear Volterra integral operator defined by

$$(\mathcal{V}\phi)(t) := \int_0^t K(t, s)\phi(s)\, ds, \quad t \in I, \tag{1.2.1}$$

where the kernel $K = K(t, s)$ is continuous on $D := \{(t, s) :\ 0 \leq s \leq t \leq T\}$. We are interested in the existence and uniqueness of solutions $u \in$

$C(I)$ of the linear integral equation

$$u(t) = f(t) + (\mathcal{V}u)(t), \quad t \in I, \tag{1.2.2}$$

when f is a given continuous function on I. In later sections we shall admit kernels $K(t, s)$ that are non-smooth or possess integrable singularities. Moreover, in Chapter 8 (Section 8.1) we shall also briefly look at the solvability of the VIE (1.2.2) in other function spaces; for example, in the space $L^2(I)$ of square-integrable functions.

As we have already mentioned in the previous section, the classical theory of linear VIEs is due to the Italian mathematician Vito Volterra. The starting point in his *Nota I* (Volterra, 1896a) is the problem of 'inverting the integral'

$$(\mathcal{V}u)(t) = g(t), \;\; t \in I, \quad \text{with } \; g(0) = 0, \tag{1.2.3}$$

in $C(I)$ when g is a given (differentiable) function. Using the terminology introduced by Lalesco (1908, p. 126), the problem is that of solving a *Volterra integral equation* of the *first kind*. Volterra showed that under certain conditions on its kernel K (see Theorem 1.4.1) the first-kind VIE (1.2.3) is equivalent to a second-kind VIE to which *Picard iteration* (introduced in Picard, 1890) can be applied. This iteration process leads, via the Neumann series associated with the kernel K in (1.2.1), to the *resolvent kernel* and hence to the *resolvent representation* of the solution u.

To be more precise, set $u_0(t) := f(t)$ and define the infinite sequence $\{u_n(t)\}$ by

$$u_n(t) := f(t) + (\mathcal{V}u_{n-1})(t), \quad t \in I, \quad n \geq 1. \tag{1.2.4}$$

Thus,

$$\begin{aligned}
u_2(t) &= f(t) + \int_0^t K(t,s)u_1(s)\,ds \\
&= f(t) + \int_0^t K(t,s)\left(f(s) + \int_0^s K(s,v)f(v)\,dv\right)ds \\
&= f(t) + \int_0^t K(t,s)f(s)\,ds + \int_0^t \left(\int_v^t K(t,s)K(s,v)\,ds\right)f(v)\,dv \\
&=: f(t) + \int_0^t K_1(t,s)f(s)\,ds + \int_0^t K_2(t,s)f(s)\,ds.
\end{aligned}$$

A straightforward induction argument then shows that the iterates $u_n(t)$ can be expressed in terms of the *iterated kernels* $K_n = K_n(t, s)$ $(n \geq 1)$, namely,

$$u_n(t) = f(t) + \int_0^t \left(\sum_{\nu=1}^{n} K_\nu(t,s)\right) f(s)\,ds, \quad n \geq 1, \tag{1.2.5}$$

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where $K_1(t,s) := K(t,s)$ and

$$K_n(t,s) := \int_s^t K_1(t,v)K_{n-1}(v,s)\,dv, \quad n \geq 2. \tag{1.2.6}$$

The iterated kernels also satisfy a relationship more general than (1.2.6), as the following result (first established in Volterra, 1896, *Nota I*, p. 316) shows.

Lemma 1.2.1 *Let $K \in C(D)$. Then for any integer r with $1 \leq r < n$ $(n \geq 2)$,*

$$K_n(t,s) = \int_s^t K_r(t,v)K_{n-r}(v,s)dv, \quad (t,s) \in D. \tag{1.2.7}$$

Proof The above assertion is obviously true for $r = 1$, since $K_1 = K$. Thus, assuming it holds for n, a simple induction argument establishes the result (1.2.7) for $n+1$. The details are left as an exercise (Exercise 1.7.1). □

Remark 1.2.2 If we associate with a given iterated kernel K_n the linear Volterra operator $\mathcal{V}_n : C(I) \to C(I)$ defined by

$$(\mathcal{V}_n\phi)(t) := \int_0^t K_n(t,s)\phi(s)ds, \quad n \geq 1,$$

then the result of Lemma 1.2.1 may be stated in a more general way, by saying that the Volterra integral operators $\mathcal{V}_n$ *commute*:

$$\mathcal{V}_r \circ \mathcal{V}_{n-r} = \mathcal{V}_{n-r} \circ \mathcal{V}_r, \quad 1 \leq r < n \ (n \geq 2). \qquad \diamond$$

Returning to (1.2.6), assuming that $K \in C(D)$ and setting

$$\bar{K} := \max\{|K(t,s)| : (t,s) \in D\},$$

it follows that the uniform bounds

$$|K_n(t,s)| \leq \bar{K}^n \frac{(t-s)^{n-1}}{(n-1)!} \leq \bar{K}^n \frac{T^{n-1}}{(n-1)!}, \quad (t,s) \in D,$$

hold for all $n \geq 1$. Thus the *Neumann series* generated by the iterated kernels K_n,

$$\sum_{n=1}^{\infty} K_n(t,s) = \lim_{\nu \to \infty} \sum_{n=1}^{\nu} K_n(t,s) =: R(t,s), \quad (t,s) \in D, \tag{1.2.8}$$

converges absolutely and uniformly on D, and this implies that its limit $R(t,s)$, the so-called *resolvent kernel* associated with the given kernel $K(t,s)$, is continuous on D. We therefore obtain

$$u(t) := \lim_{n\to\infty} u_n(t) = f(t) + \lim_{n\to\infty} \sum_{j=1}^{n} (\mathcal{V}_j f)(t)$$
$$= f(t) + \int_0^t R(t,s) f(s)\, ds, \quad t \in I.$$

We will show in Theorem 1.2.3 that this function $u \in C(I)$ is a solution of the VIE (1.2.2).

The uniform convergence of the Neumann series also implies that the resolvent kernel $R(t,s)$ satisfies the *resolvent equation*

$$R(t,s) = K(t,s) + \sum_{n=2}^{\infty} K_n(t,s) = K(t,s) + \sum_{n=2}^{\infty} \int_s^t K(t,v) K_{n-1}(v,s)\, dv,$$

which by (1.2.6) we can write as

$$R(t,s) = K(t,s) + \int_s^t K(t,v) R(v,s)\, dv, \quad (t,s) \in D. \tag{1.2.9}$$

An analogous relationship between the kernel $K(t,s)$ and its resolvent kernel $R(t,s)$ may be obtained by resorting to the result of Lemma 1.2.1 with $r = n-1$: it yields the *complementary resolvent equation*

$$R(t,s) = K(t,s) + \int_s^t R(t,v) K(v,s)\, dv, \quad (t,s) \in D. \tag{1.2.10}$$

We now have all the tools for stating the principal theorem on the existence and uniqueness of continuous solutions to linear VIEs (1.2.2). This result is due to Volterra and can be found in his *Nota I* of 1896.

Theorem 1.2.3 *Let $K \in C(D)$, and assume that R is the resolvent kernel associated with K. Then for any $f \in C(I)$ the second-kind VIE (1.2.2) possesses a unique solution $u \in C(I)$, and this solution can be written in the form*

$$u(t) = f(t) + \int_0^t R(t,s) f(s)\, ds, \quad t \in I. \tag{1.2.11}$$

Proof Replace t in the VIE (1.2.2) by v, then multiply the equation by $R(t,v)$ and integrate with respect to v over the interval $[0,t]$. Using Dirichlet's formula and the resolvent equation (1.2.10) we obtain

$$\int_0^t R(t,v) u(v)\, dv = \int_0^t R(t,v) f(v)\, dv + \int_0^t R(t,v) \left(\int_0^v K(v,s) u(s) ds \right) dv$$

$$\begin{aligned}
&= \int_0^t R(t,s)f(s)\,ds \\
&\quad + \int_0^t \left(\int_s^t R(t,v)K(v,s)\,dv\right) u(s)\,ds \\
&= \int_0^t R(t,s)f(s)\,ds + \int_0^t \big(R(t,s) - K(t,s)\big)u(s)\,ds,
\end{aligned}$$

and this shows that

$$(\mathcal{V}u)(t) = \int_0^t K(t,s)u(s)\,ds = \int_0^t R(t,s)f(s)\,ds, \quad t \in I.$$

The resolvent representation (1.2.11) for the solution of (1.2.2) now follows by substituting the above relation in (1.2.2). Thus, (1.2.11) defines a solution $u \in C(I)$ for the VIE (1.2.2).
In order to show that, under the assumptions of Theorem 1.2.3, this solution is *unique*, assume that $z \in C(I)$ is also a solution. Since

$$z(v) = f(v) + (\mathcal{V}z)(v), \quad v \in I,$$

multiplication of both sides by $R(t,v)$, integration with respect to v over $[0,t]$, and the use of Dirichlet's formula leads to

$$\begin{aligned}
\int_0^t R(t,v)z(v)\,dv &= \int_0^t R(t,v)f(v)\,dv \\
&\quad + \int_0^t \left(\int_s^t R(t,v)K(v,s)dv\right) z(s)\,ds \\
&= \int_0^t R(t,v)f(v)\,dv + \int_0^t \big(R(t,s) - K(t,s)\big)z(s)\,ds.
\end{aligned}$$

Here, we have employed the second resolvent equation (1.2.10). Since u and z are continuous solutions of (1.2.2), the above equation reduces to

$$u(t) - f(t) - \int_0^t K(t,s)z(s)\,ds = \big(u(t) - f(t)\big) - \big(z(t) - f(t)\big) = 0, \quad t \in I.$$

The uniqueness of the solution u given by (1.2.11) can also be established directly via *Gronwall's Lemma* (cf. Lemma 1.2.7 in Section 1.2.5). If u and z are two (continuous) solutions of (1.2.2) it follows that

$$u(t) - z(t) = \big(\mathcal{V}(u - z)\big)(t), \quad t \in I.$$

Hence, assuming again that $|K(t,s)| \le \bar{K}$ in D, we obtain

$$|u(t) - z(t)| \le \bar{K} \int_0^t |u(s) - z(s)|ds, \quad t \in I.$$

Lemma 1.2.7 and the continuity of u and z thus imply that

$$|u(t) - z(t)| \leq 0 \cdot \exp(\bar{K}t) = 0 \quad \text{for all} \;\; t \in I,$$

and this yields $u(t) = z(t)$ for all $t \in I$, as asserted. □

Remark 1.2.4 The Volterra integral equation

$$u(t) = f(t) + \lambda(\mathcal{V}u)(t), \quad t \in I,$$

possesses a unique solution $u \in C(I)$ for *any* (real or complex) parameter λ. This follows directly from the form of the Neumann series corresponding to the kernel $\lambda K(t, s)$, since the nth iterated kernel is now bounded on D by $|\lambda|^n \bar{K}^n T^{n-1}/(n-1)!$. Hence, the inverse of the linear operator $\mathcal{I} - \lambda\mathcal{V}$ exists as a bounded linear operator on $C(I)$ for any kernel $K \in C(D)$ and any (real or complex) number λ. In other words, the *spectrum* $\sigma(\mathcal{V})$ of $\mathcal{V}$ (that is, the set of values λ for which the operator $\mathcal{I} - \lambda\mathcal{V}$ is not invertible in $C(I)$) is given by $\{0\}$. (As we shall see in Section 8.2, an operator possessing this property is called *quasi-nilpotent*.) Thus, for every $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$) the VIE $u = f + \lambda\mathcal{V}u$ possesses a unique solution $u \in C(I)$ for any $f \in C(I)$, and this solution is given by

$$u = (\mathcal{I} - \lambda\mathcal{V})^{-1} f = (\mathcal{I} + \lambda\mathcal{R}) f,$$

where $\mathcal{R} : \; C(I) \to C(I)$ is the *resolvent operator* corresponding to $\mathcal{V}$,

$$(\mathcal{R}f)(t) := \int_0^t R(t, s) f(s)\, ds, \;\; t \in I.$$

We shall present a more detailed study of the properties of resolvent kernels and resolvent operators in Sections 6.2 and 8.3. ◇

Remark 1.2.5 The above observations are in general not true for the *Fredholm integral operator*

$$(\mathcal{F}u)(t) := \int_0^T K(t, s) u(s)\, ds, \;\; t \in I,$$

with $K \in C(I \times I)$, and the corresponding Fredholm integral equation of the second kind,

$$u(t) = f(t) + \lambda(\mathcal{F}u)(t), \quad t \in I.$$

Here, the operator $\mathcal{I} - \lambda\mathcal{F}$ may not be invertible for all $\lambda \in \mathbb{C}$ (compare e.g. Fredholm, 1903; Cochran, 1972; Gohberg & Goldberg, 1980; or Kress, 1999; see also Appendix A.2). A simple example is the (finite-rank) Fredholm integral operator $\mathcal{F}$ with kernel $K(t, s) = a(t)b(s)$ $(a, b \in C(I))$ and $I = [0, 1]$.

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It is readily verified (by setting $c_1 := \int_0^1 b(s)u(s)ds$) that $\mathcal{I} - \lambda\mathcal{F}$ is invertible if, and only if, λ is such that

$$1 - \lambda \int_0^1 a(s)b(s)\,ds \neq 0. \qquad \diamond$$

Theorem 1.2.3 allows us to establish an analogous existence and solution representation result for the linear Volterra *integro-differential equation* (VIDE)

$$u'(t) = a(t)u(t) + f(t) + (\mathcal{V}u)(t), \ \ t \in I, \ \ \text{with} \ \ u(0) = u_0. \qquad (1.2.12)$$

Theorem 1.2.6 *If $a,\ f \in C(I)$ and $K \in C(D)$, the VIDE (1.2.12) possesses a unique solution $u \in C^1(I)$ for every initial value u_0. This solution can be written in the form*

$$u(t) = S(t,0)u_0 + \int_0^t S(t,s)f(s)\,ds, \ \ t \in I,$$

where the (differential) resolvent kernel S (defined in the proof below) lies in $C^1(D)$.

Proof The initial-value problem (1.2.12) is equivalent to the VIE

$$u(t) = f_0(t) + \int_0^t K_0(t,s)u(s)\,ds, \ \ t \in I.$$

Since the functions

$$f_0(t) := u_0 + \int_0^t f(s)\,ds \quad \text{and} \quad K_0(t,s) := a(s) + \int_s^t K(v,s)\,dv$$

satisfy $f_0 \in C^1(I)$ and $\partial K_0/\partial t \in C(D)$, the resolvent kernel $R_0(t,s)$ of $K_0(t,s)$ is also in $C^1(D)$. By Theorem 1.2.3 the solution of this VIE is

$$u(t) = f_0(t) + \int_0^t R_0(t,s)f_0(s)\,ds, \ \ t \in I;$$

it can be written as

$$u(t) = \Big(1 + \int_0^t R_0(t,s)\,ds\Big)u_0 + \int_0^t \Big(1 + \int_s^t R_0(t,v)\,dv\Big)f(s)\,ds.$$

Setting

$$S(t,s) := 1 + \int_s^t R_0(t,v)\,dv, \ \ (t,s) \in D,$$

we obtain the statement of Theorem 1.2.6, since $S \in C^1(D)$. □

If the given function f in the VIE (1.2.2) is in $C^1(I)$, it is often advantageous (for example, when studying the asymptotic stability of solutions; see Chapter 6) to represent the solution of this VIE in a form resembling the familiar *variation-of-constants formula* for a linear first-order ordinary differential equation. In order to derive this alternative representation, we first observe that the special VIE

$$u(t) = f(t) + \int_0^t a(s)u(s)\,ds, \quad t \in I, \tag{1.2.13}$$

with $f \in C^1(I)$ and $a \in C(I)$, is equivalent to the initial-value problem

$$u'(t) = a(t)u(t) + f'(t), \quad t \in I, \quad u(0) = f(0), \tag{1.2.14}$$

whose solution is given by

$$u(t) = \exp\left(\int_0^t a(v)dv\right) f(0) + \int_0^t \exp\left(\int_s^t a(v)\,dv\right) f'(s)\,ds.$$

Setting

$$Q(t,s) := \exp\left(\int_s^t a(v)dv\right), \quad (t,s) \in D,$$

we obtain the well-known representation of the solution of (1.2.14), namely

$$u(t) = Q(t,0)u(0) + \int_0^t Q(t,s)f'(s)\,ds, \quad t \in I. \tag{1.2.15}$$

On the other hand, we have seen that the resolvent kernel $R(t,s)$ associated with the kernel $K(t,s) = a(s)$ in (1.2.13) satisfies the resolvent equation (1.2.9),

$$R(t,s) = a(s) + \int_s^t a(v)R(v,s)\,dv, \quad (t,s) \in D, \tag{1.2.16}$$

and hence it solves the initial-value problem

$$\frac{\partial R(t,s)}{\partial t} = a(t)R(t,s), \quad R(s,s) = a(s), \quad s \in I. \tag{1.2.17}$$

Its unique solution is

$$R(t,s) = a(s)\exp\left(\int_s^t a(v)\,dv\right), \quad (t,s) \in D.$$

In other words, we have shown that for the *special* Volterra integral equation (1.2.13),

$$\frac{\partial Q(t,s)}{\partial s} = -R(t,s), \quad (t,s) \in D. \tag{1.2.18}$$

We will now prove that the *variation-of-constants formula* (1.2.15) can be extended to the general linear VIE (1.2.2) (compare also Bownds & Cushing, 1973 and Brunner & van der Houwen, 1986, pp. 13–14).

Theorem 1.2.7 *Assume that $f \in C^1(I)$ and $K \in C(D)$. Then the (unique) solution $u \in C(I)$ of the VIE*

$$u(t) = f(t) + \int_0^t K(t,s)u(s)\,ds, \quad t \in I$$

is given by the variation-of-constants formula

$$u(t) = S(t,0)f(0) + \int_0^t S(t,s)f'(s)\,ds, \quad t \in I, \tag{1.2.19}$$

where $S(t,s)$ is the (unique) continuous solution of

$$S(t,s) = 1 + \int_s^t K(t,v)S(v,s)\,dv, \quad (t,s) \in D. \tag{1.2.20}$$

Moreover, $S(t,s)$ is related to the resolvent kernel $R(t,s)$ of the given kernel $K(t,s)$ by

$$-\frac{\partial S(t,s)}{\partial s} = R(t,s), \quad (t,s) \in D. \tag{1.2.21}$$

Proof It follows from the definition (1.2.20) of $S(t,s)$ and the continuity of $K(t,s)$ that $S(t,t) = 1$ for $t \in I$. Using integration by parts on the right-hand side of (1.2.19) we obtain

$$S(t,0)f(0) + \Big(f(t) - S(t,0)f(0) - \int_0^t \frac{\partial S(t,s)}{\partial s} f(s)\,ds\Big)$$
$$= f(t) - \int_0^t \frac{\partial S(t,s)}{\partial s} f(s)\,ds\ .$$

Thus, we may write (1.2.19) in the form

$$u(t) = f(t) - \int_0^t \frac{\partial S(t,s)}{\partial s} f(s)\,ds, \quad t \in I. \tag{1.2.22}$$

Since u is the unique continuous solution of the VIE (1.2.2), comparison of (1.2.11) and (1.2.22) shows that the relation (1.2.21) between $S(t,s)$ and $R(t,s)$ in (1.2.11) is valid, and hence the solution representation (1.2.19) is true. □

We conclude this section by observing that for certain classes of linear VIEs it is not necessary to resort to Volterra's classical resolvent kernel approach to establish the existence and uniqueness of continuous solutions. An important