

Cambridge University Press

978-1-107-09761-2 — Discriminant Equations in Diophantine Number Theory

Jan-Hendrik Evertse , Kálmán Győry

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