# Introduction

Clifford algebras find their use in many areas of mathematics: in differential analysis, where operators of Dirac type are used in proofs of the Atiyah-Singer index theorem, in harmonic analysis, where the Riesz transforms provide a higher-dimensional generalization of the Hilbert transform, in geometry, where spin groups illuminate the structure of the classical groups, and in mathematical physics, where Clifford algebras provide a setting for electromagnetic theory, spin 1/2 particles, and the Dirac operator in relativistic quantum mechanics. This book is intended as a straightforward introduction to Clifford algebras, without going on to study any of the above topics in detail (suggestions for further reading are made at the end). This means that it concentrates on the underlying structure of Clifford algebras, and this inevitably means that it approaches the subject algebraically.

The first part is concerned with the background from algebra that is required. The first chapter describes, without giving details, the necessary knowledge of groups and vector spaces that is needed. Any reader who is not familiar with this material should consult standard texts on algebra, such as Mac Lane and Birkhoff [MaB], Jacobson [Jac] or Cohn [Coh]. Otherwise, skim through it, to familiarize yourself with the notation and terminology that is used.

The second chapter deals with algebras, and modules over algebras. It turns out that the algebra $\mathbf{H}$ of quaternions has an important part to play in the theory of Clifford algebras, and fundamental properties of this algebra are developed here. It also turns out that the Clifford algebras that we study are isomorphic to the algebra of $\mathbf{D}$-endomorphisms of $\mathbf{D}^k$, where $\mathbf{D}$ is either the real field $\mathbf{R}$, the complex field $\mathbf{C}$ or the algebra $\mathbf{H}$; we develop the theory of modules over an algebra far enough to prove Wedderburn's theorem, which explains why this is the case.

Tensor products of various forms are an invaluable tool for constructing Clifford algebras. Many mathematicians are uncomfortable with tensor products; in Chapter 3 we provide a careful account of the multilinear algebra that is needed. This involves finite-dimensional vector spaces, where there is a powerful and effective duality theory, and we make unashamed use of this duality to construct the spaces of tensor products that we need.

The second part is the heart of this book. Clifford algebras are constructed, starting from a vector space equipped with a quadratic form. Chapter 4 is concerned with quadratic forms on finite-dimensional real vector spaces. The reader is probably familiar with the special case of Euclidean space, where the quadratic form is positive definite. The general case is, perhaps surprisingly, considerably more complicated, and we provide complete details. In particular, we prove the Cartan-Dieudonné theorem, which shows that an isometry of a regular quadratic space is the product of simple reflections.

In Chapter 5, we begin the study of Clifford algebras. This can be done at different levels of generality. At one extreme, we could consider Clifford algebras over an arbitrary field; these are important, for example in number theory, but this is too general for our purposes. At the other extreme, we could restrict attention to Clifford algebras over the complex field **C**; this has the advantage that **C** is algebraically complete, which leads to considerable simplifications, but in the process it removes many interesting ideas from consideration. In fact, we shall consider Clifford algebras over the real field; this provides enough generality to consider many important ideas, while at the same time provides an appropriate setting for differential analysis, harmonic analysis and mathematical physics. We shall see that the complex field **C** is an example of such a Clifford algebra; one of its salient features is the conjugation involution. Universal Clifford algebras also admit such an involutory automorphism, the *principal involution*. This leads to a $\mathbf{Z}_2$ grading of such algebras; they are *super-algebras*. This is one of the fundamental features of the structure of Clifford algebras. But they are more complicated than that; besides the principal involution there are two important involutory anti-automorphisms.

Much of the charm of Clifford algebras lies in the fact that they have interesting concrete representations, and in Chapters 6 and 7 we calculate many of these. In particular, we use arguments involving Clifford algebras to prove Frobenius' theorem, that the only finite-dimensional real division algebras are the real field **R**, the complex field **C** and the algebra **H** of quaternions. The calculations involve dimensions up to 5; we also establish a partial periodicity of order 4, and Cartan's periodicity theorem, with periodicity of order 8, which enable all Clifford algebras to be calculated. As a result of these calculations, we find that every Clifford algebra is either isomorphic to a full matrix algebra over **R**, **C** or **H**, or is isomorphic to the direct sum of two copies of one of these. This is a consequence of Wedderburn's theorem: a finite-dimensional simple real algebra is isomorphic to a full matrix algebra over a finite-dimensional real division algebra. Matrices act on vector spaces; at the end of Chapter 7 we introduce *spinor spaces*, which are vector spaces on which a Clifford algebra acts irreducibly.

A large part of mathematics is concerned with symmetry, and the orthogonal group and special orthogonal group describe the linear symmetries of regular quadratic spaces. An important feature of Clifford algebras is that the group of invertible elements of a Clifford algebra contains a subgroup, the *spin group*, which provides a double cover of the corresponding special orthogonal group. In Chapter 8, spin groups are defined, and their basic properties are proved. Spin groups, and their actions, are calculated for spaces up to dimension 4, and for 5- and 6-dimensional Euclidean space.

In the third part, we describe some of the applications of Clifford algebras. Our intention here is to provide an introduction to a varied collection of applications; to whet the appetite, so that the reader will wish to pursue his or her interests further.

A great deal of interest in Clifford algebras goes back to 1927 and 1928. In 1927, Pauli introduced the so-called Pauli spin matrices to provide a quantum mechanical framework for particles with spin 1/2, and in 1928, Dirac introduced the Dirac operator (though not with this name, nor in terms of a Clifford algebra) to construct the Dirac equation, which describes the relativistic behaviour of an electron. In Chapter 9, we describe the use of the Pauli spin matrices to represent the angular momentum of particles with spin 1/2. We introduce the Dirac operator, and construct the Dirac equation. We also show that Maxwell's equations for electromagnetic fields can be expressed as a single equation involving the Dirac operator.

Clifford algebras have important applications in differential and harmonic analysis. A fascinating topic in two dimensions is the relationship between harmonic functions and analytic functions, using the Hilbert transform. In Chapter 10, we show how the Dirac operator, and an augmented Dirac operator, can be used to extend the idea of analyticity to higher dimensions, so that corresponding problems can be considered there; the Hilbert transform is replaced by the system of Riesz transforms. As a particular application, we show how to extend to higher dimensions the celebrated theorem of the brothers Riesz, which shows that if the harmonic Dirichlet extension of a complex measure is analytic, then the measure is absolutely continuous with respect to Lebesgue measure, and is represented by a function in the Hardy space $H_1$.

These results concern functions defined on half a vector space. An even more important use of Dirac operators concerns analysis on a compact Riemannian manifold; this leads to proofs of the Atiyah-Singer index theorem. A full account of this demands a detailed knowledge of Riemannian geometry, and so cannot be given at this introductory level, but we end Chapter 10 by giving a brief description of the set-up in which Dirac operators can be defined.

The spin groups provide a double cover of the special orthogonal groups. In Chapter 11, we show how this can be used to describe the irreducible representations of the special orthogonal groups of Euclidean spaces of dimensions 2, 3 and 4. The use of the double cover goes much further, but again this requires a detailed understanding of the representation theory of compact Lie groups, inappropriate for a book at this elementary level.

These remarks show that this book is only an introduction to a large subject, with many applications. In the final chapter, we make some further comments, and also make some suggestions for further reading.

I would especially like to thank the referees for their dissatisfaction with earlier drafts of this book, which led to improvements both of content and of presentation. I have worked hard to remove errors, but undoubtedly some remain. Corrections and further comments can be found on my personal web-page at `www.dpmms.cam.ac.uk`.

I acknowledge the use of Paul Taylor's 'diagrams' package, which I used for the commutative diagrams in the text; I found the package easy to use.

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# PART ONE

# THE ALGEBRAIC ENVIRONMENT

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# 1

# Groups and vector spaces

The material in this chapter should be familiar to the reader, but it is worth reading through it to become familiar with the notation and terminology that is used. We shall not give details; these are given in standard textbooks, such as Mac Lane and Birkhoff [MaB], Jacobson [Jac] or Cohn [Coh].

## 1.1 Groups

A *group* is a non-empty set $G$ together with a *law of composition*, a mapping $(g, h) \to gh$ from $G \times G$ to $G$, which satisfies:

1. $(gh)j = g(hj)$ for all $g, h, j$ in $G$ (associativity),
2. there exists $e$ in $G$ such that $eg = ge = g$ for all $g \in G$, and
3. for each $g \in G$ there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

It then follows that $e$, the *identity element*, is unique, and that for each $g \in G$ the *inverse* $g^{-1}$ is unique.

A group $G$ is *abelian*, or *commutative*, if $gh = hg$ for all $g, h \in G$. If $G$ is abelian, then the law of composition is often written as addition: $(g, h) \to g + h$. In such a case, the identity is denoted by 0, and the inverse of $g$ by $-g$.

A non-empty subset $H$ of a group $G$ is a *subgroup* of $G$ if $h_1h_2 \in H$ whenever $h_1, h_2 \in H$, and $h^{-1} \in H$ whenever $h \in H$. $H$ then becomes a group under the law of composition inherited from $G$.

If $A$ is a subset of a group $G$, there is a smallest subgroup $\mathrm{Gp}(A)$ of $G$ which contains $A$, the *subgroup generated by* $A$. If $A = \{g\}$ is a singleton then we write $\mathrm{Gp}(g)$ for $\mathrm{Gp}(A)$. Then $\mathrm{Gp}(g) = \{g^n : n \text{ an integer}\}$, where $g^0 = e$, $g^n$ is the product of $n$ copies of $g$ when $n > 0$, and $g^n$

is the product of $|n|$ copies of $g^{-1}$ when $n < 0$. A group $G$ is *cyclic* if $G = \mathrm{Gp}(g)$ for some $g \in G$.

If $G$ has finitely many elements, then the *order* $o(G)$ of $G$ is the number of elements of $G$. If $G$ has infinitely many elements, then we set $o(G) = \infty$. If $g \in G$ then the *order* $o(g)$ of $g$ is the order of the group $\mathrm{Gp}(g)$.

A mapping $\theta : G \to H$ from a group $G$ to a group $H$ is a *homomorphism* if $\theta(g_1g_2) = \theta(g_1)\theta(g_2)$, for $g_1, g_2 \in G$. It then follows that $\theta$ maps the identity in $G$ to the identity in $H$, and that $\theta(g^{-1}) = (\theta(g))^{-1}$, for $g \in G$. A bijective homomorphism is called an *isomorphism*, and an isomorphism $G \to G$ is called an *automorphism* of $G$. The set $\mathrm{Aut}(G)$ of automorphisms of $G$ forms a group, when composition of mappings is taken as the group law of composition.

A subgroup $K$ of a group $G$ is a *normal*, or *self-conjugate*, subgroup if $g^{-1}hg \in K$ for all $g \in G$ and $k \in K$. If $\theta : G \to H$ is a homomorphism, then the *kernel* $\ker(\theta)$ of $\theta$, the set $\{g \in G : \theta g = e_H\}$ (where $e_H$ is the identity in $H$) is a normal subgroup of $G$. Conversely, suppose that $K$ is a normal subgroup of $G$. The relation $g_1 \sim g_2$ on $G$ defined by setting $g_1 \sim g_2$ if $g_1^{-1}g_2 \in K$ is an equivalence relation on $G$. The equivalence classes are called the *cosets* of $K$ in $G$. If $C$ is a coset of $K$ then $C$ is of the form $Kg = \{kg : k \in K\}$ and $Kg = gK = \{gk : k \in K\}$. If $C_1$ and $C_2$ are cosets of $K$ in $G$ then so is $C_1C_2 = \{c_1c_2 : c_1 \in C_1, c_2 \in C_2\}$; if $C_1 = Kg_1$ and $C_2 = Kg_2$ then $C_1C_2 = Kg_1g_2$. With this law of composition, the set $G/K$ of cosets becomes a group, the *quotient group*. The identity is $K$ and $(Kg)^{-1} = Kg^{-1}$. The quotient mapping $q : G \to G/K$ defined by the equivalence relation is then a homomorphism of $G$ onto $G/K$, with kernel $K$, and $q(g) = Kg$.

A group $G$ is *simple* if it has no normal subgroups other than $\{e\}$ and $G$.

We denote the group with one element by 1, or, if we are denoting composition by addition, by 0. Suppose that $(G_0 = 1, G_1, \ldots, G_k, G_{k+1} = 1)$ is a sequence of groups, and that $\theta_j : G_j \to G_{j+1}$ is a homomorphism, for $0 \le j \le k$. Then the diagram

$$1 \xrightarrow{\theta_0} G_1 \xrightarrow{\theta_1} G_2 \xrightarrow{\theta_2} \ldots \xrightarrow{\theta_{k-1}} G_k \xrightarrow{\theta_k} 1$$

is an *exact sequence* if $\theta_{j-1}(G_{j-1})$ is the kernel of $\theta_j$, for $1 \le j \le k$. If $k = 3$, the sequence is a *short exact sequence*. For example, if $K$ is a normal subgroup of $g$ and $q : G \to G/K$ is the quotient mapping, then

$$1 \longrightarrow K \xrightarrow{\subseteq} G \xrightarrow{q} G/K \longrightarrow 1$$

is a short exact sequence. If $A$ is a subset of a group $G$ then the *centralizer* $C_G(A)$ of $A$ in $G$, defined as

$$C_G(A) = \{g \in G : ga = ag \text{ for all } a \in A\},$$

is a subgroup of $G$. The *centre* $Z(G)$, defined as

$$Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\},$$

(which is $C_G(G)$), is a normal subgroup of $G$.

The product of two groups $G_1 \times G_2$ is a group, when composition is defined by

$$(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2)$$

We identify $G_1$ with the subgroup $G_1 \times \{e_2\}$ and $G_2$ with the subgroup $\{e_1\} \times G_2$.

Let us now list some of the groups that we shall meet later.

1. The real numbers $\mathbf{R}$ form an abelian group under addition. The set $\mathbf{Z}$ of integers is a subgroup of $\mathbf{R}$. The set $\mathbf{R}^*$ of non-zero real numbers is a group under multiplication.
2. Any two groups of order 2 are isomorphic. We shall denote the multiplicative subgroup $\{1, -1\}$ of $\mathbf{R}^*$ by $D_2$, and the additive group $\{0, 1\}$ by $\mathbf{Z}_2$. $\mathbf{Z}_2$ is isomorphic to the quotient group $\mathbf{Z}/2\mathbf{Z}$. Small though they are, these groups of order 2 play a fundamental role in the theory of Clifford algebras (and in many other branches of mathematics and physics).

   Suppose that we have a short exact sequence

   $$1 \longrightarrow D_2 \xrightarrow{j} G_1 \xrightarrow{\theta} G_2 \longrightarrow 1.$$

   Then $j(D_2)$ is a normal subgroup of $G_1$, from which it follows that $j(D_2)$ is contained in the centre of $G_1$. If $g \in G_1$ then we write $-g$ for $j(-1)g$. Then $\theta(g) = \theta(-g)$, and if $h \in G_2$ then $\theta^{-1}\{h\} = \{g, -g\}$ for some $g$ in $G$. In this case, we say that $G_1$ is a *double cover* of $G_2$. Double covers play a fundamental role in the theory of spin groups; these are considered in Chapter 8.
3. A bijective mapping of a set $X$ onto itself is called a *permutation*. The set $\Sigma_X$ of permutations of $X$ is a group under the composition of mappings. $\Sigma_X$ is not abelian if $X$ has at least three elements. We

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denote the group of permutations of the set $\{1, \dots, n\}$ by $\Sigma_n$. $\Sigma_n$ has order $n!$. A *transposition* is a permutation which fixes all but 2 elements. $\Sigma_n$ has a normal subgroup $A_n$ of order $n!/2$, consisting of those permutations that can be expressed as the product of an even number of transpositions. Thus we have a short exact sequence

$$1 \longrightarrow A_n \xrightarrow{\subseteq} \Sigma_n \xrightarrow{\epsilon} D_2 \longrightarrow 1.$$

If $\sigma \in \Sigma_n$ then $\epsilon(\sigma)$ is the *signature* of $\sigma$; $\epsilon(\sigma) = 1$ if $\sigma \in A_n$, and $\epsilon(\sigma) = -1$ otherwise.

4. The complex numbers $\mathbf{C}$ form an abelian group under addition, and $\mathbf{R}$ can be identified as a subgroup of $\mathbf{C}$. The set $\mathbf{C}^*$ of non-zero complex numbers is a group under multiplication. The set $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ is a subgroup of $\mathbf{C}^*$. There is a short exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\subseteq} \mathbf{R} \xrightarrow{q} \mathbf{T} \longrightarrow 1$$

where $\mathbf{Z}$ is the additive group of integers and $q(\theta) = e^{2\pi i\theta}$.

5. The subset $\mathbf{T}_n = \{e^{2\pi ij/n} : 0 \le j < n\} = \{z \in \mathbf{C} : z^n = 1\}$ of $\mathbf{T}$ is a cyclic subgroup of $\mathbf{T}$ of order $n$. Conversely, if $G = Gp(g)$ is a cyclic group of order $n$ then the mapping $g^k \to e^{2\pi ik/n}$ is an isomorphism of $G$ onto $\mathbf{T}_n$.

6. Let $D$ denote the group of isometries of the complex plane $\mathbf{C}$ which fix the origin:

$$D = \{g : \mathbf{C} \to \mathbf{C} : g(0) = 0 \text{ and } |g(z) - g(w)| = |z - w| \text{ for } z, w \in \mathbf{C}\}.$$

$D$ is the *full dihedral group*. Then an element of $D$ is either a *rotation* $R_\theta$ (where $R_\theta(z) = e^{i\theta}z$) or a *reflection* $S_\theta$ (where $S_\theta(z) = e^{i\theta}\bar{z}$). The set *Rot* of rotations is a subgroup of $D$, and the mapping $R : e^{i\theta} \to R_\theta$ is an isomorphism of $\mathbf{T}$ onto *Rot*. In particular, $R_\pi(z) = -z$. Since

$$S_\theta^2(z) = e^{i\theta}\overline{(e^{i\theta}\bar{z})} = e^{i\theta}e^{-i\theta}z = z,$$

$S_\theta^2$ is the identity. A similar calculation shows that $S_\theta^{-1}R_\phi S_\theta = R_{-\phi}$, so that $R$ is a normal subgroup of $D$. We have an exact sequence

$$1 \longrightarrow \mathbf{T} \xrightarrow{R} D \xrightarrow{\delta} D_2 \longrightarrow 1$$

where $\delta(R_\theta)) = 1$ and $\delta(S_\theta) = -1$, for $\theta \in [0, 2\pi)$.

7. If $n \ge 2$, let $R_n = R(\mathbf{T}_n)$ and let $D_{2n} = R_n \cup R_n S_0$. $D_{2n}$ is a subgroup of $D$, called the *dihedral group of order* $2n$. (Warning: some authors denote this group by $D_n$.) The group $D_4 \cong D_2 \times D_2$, and so $D_4$ is

abelian. If $n \geq 3$ then $D_{2n}$ is the group of symmetries of a regular polygon with $n$ vertices, with centre the origin; $D_{2n}$ is a non-abelian subgroup of $D$ of order $2n$. If $n = 2k$ is even, then $Z(D_{2n}) = \{1, r_{-1}\}$, and we have a short exact sequence

$$1 \longrightarrow D_2 \longrightarrow D_{2n} \longrightarrow D_{2k} \longrightarrow 1;$$

$D_{2n}$ is a double cover of $D_{2k}$. If $n = 2k+1$ is odd, then $Z(D_{2n}) = \{1\}$.

8. In particular, the dihedral group $D_8$ is a non-abelian group, which is the group of symmetries of a square with centre the origin. Let us set $\alpha = r_i$, $\beta = \sigma_1$ and $\gamma = \sigma_i$. Then $D_8 = \{\pm 1, \pm\alpha, \pm\beta, \pm\gamma\}$ (where $-x$ denotes $r_{-1}x = xr_{-1}$), and

$$\begin{array}{ccc} \alpha\beta = \gamma & \beta\gamma = \alpha & \gamma\alpha = \beta \\ \beta\alpha = -\gamma & \gamma\beta = -\alpha & \alpha\gamma = -\beta \\ \alpha^2 = -1 & \beta^2 = 1 & \gamma^2 = 1. \end{array}$$

There is a short exact sequence

$$1 \longrightarrow D_2 \longrightarrow D_8 \longrightarrow D_4 \longrightarrow 1.$$

9. The *quaternionic group* $\mathcal{Q}$ is a group of order 8, with elements $\{\pm 1, \pm\boldsymbol{i}, \pm\boldsymbol{j}, \pm\boldsymbol{k}\}$, with identity element 1, and law of composition defined by

$$\begin{array}{ccc} \boldsymbol{ij} = \boldsymbol{k} & \boldsymbol{jk} = \boldsymbol{i} & \boldsymbol{ki} = \boldsymbol{j} \\ \boldsymbol{ji} = -\boldsymbol{k} & \boldsymbol{kj} = -\boldsymbol{i} & \boldsymbol{ik} = -\boldsymbol{j} \\ \boldsymbol{i}^2 = -1 & \boldsymbol{j}^2 = -1 & \boldsymbol{k}^2 = -1, \end{array}$$

and $(-1)x = x(-1) = -x$, $(-1)(-x) = (-x)(-1) = x$ for $x = 1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. Then $Z(\mathcal{Q}) = \{1, -1\}$, and there is a short exact sequence

$$1 \longrightarrow D_2 \longrightarrow \mathcal{Q} \longrightarrow D_4 \longrightarrow 1;$$

$\mathcal{Q}$ is a double cover of $D_4$.

The groups $D_8$ and $\mathcal{Q}$ are of particular importance in the study of Clifford algebras. Although they both provide double covers of $D_4$, they are not isomorphic. $\mathcal{Q}$ has six elements of order 4, one of order 2 and one of order 1. $D_8$ has two elements of order 4, five of order 2 and one of order 1.