Eigenvalues, Multiplicities and Graphs

The arrangement of nonzero entries of a matrix, described by the graph of the matrix, limits the possible geometric multiplicities of the eigenvalues, which are far more limited by this information than are algebraic multiplicities or the numerical values of the eigenvalues. This book gives a unified development of how the graph of a symmetric matrix influences the possible multiplicities of its eigenvalues. While the theory is richest in cases where the graph is a tree, work on eigenvalues, multiplicities and graphs has provided the opportunity to identify which ideas have analogs for non-trees, and those for which trees are essential. It gathers and organizes the fundamental ideas to allow students and researchers to easily access and investigate the many interesting questions in the subject.

CHARLES R. JOHNSON is Class of 1961 Professor of Mathematics at College of William and Mary. He is the recognized expert in the interplay between linear algebra and combinatorics, as well as many parts of matrix analysis. He is coauthor of *Matrix Analysis, Topics in Matrix Analysis* (both with Roger Horn), and *Totally Nonnegative Matrices* (with Shaun Fallat).

CARLOS M. SAIAGO is Assistant Professor of Mathematics at Universidade Nova de Lisboa, and is the author of 15 papers on eigenvalues, multiplicities and graphs.

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CHARLES R. JOHNSON
College of William and Mary, Williamsburg, Virginia

CARLOS M. SAIAGO
Universidade Nova de Lisboa

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It is a pleasure to dedicate this work to my precious family, Dana, Jennifer, Emily, Simon, Cora, Maeve, Chad and Chris, and to my many students whose enjoyment in thinking about this subject has led to many valuable insights reported herein.

The fond memory of my brother Tom and my sister Pat will always be with me.

Charles R. Johnson

To my wonderful parents and my precious sister.

Carlos M. Saiago

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Preface

Among the n eigenvalues of an n-by-n matrix may be several repetitions (the number of which counts toward the total of n). For general matrices over a general field, these multiplicities may be algebraic (the number of appearances as a root of the characteristic polynomial) or geometric (the dimension of the corresponding eigenspace). These multiplicities are quite important in the analysis of matrix structure because of numerical calculation, a variety of applications, and for theoretical interest. We are primarily concerned with geometric multiplicities and, in particular but not exclusively, with real symmetric or complex Hermitian matrices, for which the two notions of multiplicity coincide.

It has been known for some time, and is not surprising, that the arrangement of nonzero entries of a matrix, conveniently described by the graph of the matrix, limits the possible geometric multiplicities of the eigenvalues. Much less limited by this information are either the algebraic multiplicities or the numerical values of the (distinct) eigenvalues. So, it is natural to study exactly how the graph of a matrix limits the possible geometric eigenvalue multiplicities.

Organized study of "eigenvalues, multiplicities and graphs" really began in the 1990s, though two earlier papers, [P] and [Wie], play an important role, including motivational. There had also been considerable interest in the eigenvalues of particular matrices with a given graph, such as the adjacency or Laplacian matrix. It was recognized early that the theory is most rich in case the graph is minimally connected, i.e., a tree. For this reason, the theory is relatively well developed for trees. However, in recent papers and in the preparation of this monograph, there has been an opportunity to identify more clearly which ideas have analogs for nontrees and for which ideas trees are essential. We have also recently noticed that for trees, and sometimes for general graphs, ideas about real symmetric/complex Hermitian matrices carry over to geometric multiplicities in general matrices over a field, sometimes under a diagonalizability hypothesis. This is an important advance that we have included herein (Chapter 12); the proofs are necessarily very different, and we have also included earlier proofs for the symmetric case, which are of interest for themselves and support other work.

We include briefly in Chapter 0 some necessary background, with which some readers may not be familiar. After an introduction with a detailed problem description and supporting ideas, the most basic theory of multiplicities for trees is given in complete detail in Chapter 2, with some elaboration in Chapter 4, including the role of eigenvectors. Chapters 3 and 5 give the theory of maximum multiplicity and related ideas. What is known about the minimum number of distinct eigenvalues in a symmetric matrix whose graph is a tree is presented in Chapter 6. The difficult problem of constructing the multiplicity lists that do occur is addressed in Chapter 7, with the several techniques that are known. In Chapters 8, 9 and 10, these are used to describe all possible multiplicity lists for certain large classes of trees. In Chapter 10, the rather new and extraordinarily important idea of linear trees is discussed. For linear trees, several natural conjectures that failed for general trees turn out to be true. Chapter 11 discusses several particular results about multiplicity lists for nontrees. In Chapter 12, the new ideas about geometric multiplicities in general matrices over a field are given. They strongly parallel the symmetric case, with a few major exceptions.

Considerable useful information is given in several appendixes. All multiplicity lists for all the more than 430 trees on < 12 vertices are given. This information is also available in a queriable electronic database that the authors and students of C. Johnson have put together over the years. The 12 vertex trees are included there as well, providing a powerful research tool. Appendixes also include the seeds of diameter < 8 used for minimizing the number of distinct eigenvalues and the unfoldings for which the minimum number exceeds the diameter.

Our purpose has been to gather together, in one convenient place, the most fundamental ideas in this area, and examples of how they may be used, in hopes that it will be easier for students, and other researchers new to the area, to get started on the many interesting questions in the subject. These ideas should be what is needed to support future research. It was not practical or productive to try to include everything that has been done on the topic. Some important work is summarized and referenced without complete proofs. Of course, any omissions may leave out something useful, but we have tried to be inclusive of interesting results in our bibliography, which includes much work not covered in the text.

An extensive bibliography of papers and related books, to date, in this and related areas is given (and often referred to), as well as an index and a list of terms and symbols used.

The problem of describing all possible multiplicity lists for any tree (let alone graph) is still not solved, and we hope that the appearance of this monograph will be a useful step toward its resolution.

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This work was carried out within the activities of Centro de Matemática e Aplicações da Faculdade de Ciências e Tecnologia da Universidade NOVA de Lisboa (CMA/FCT/UNL), and it was partially supported by Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013.

List of Terms and Symbols

• CHAPTER 0

$\mathbb{R}$	the real numbers, 1
$\mathbb{C}$	the complex numbers, 1
m-by-n	m rows and n columns, 1
$\mathcal{M}_n(\mathbb{C})$	n-by-n matrices over $\mathbb{C}$, 1
$\mathcal{M}_n(\mathbb{R})$	n-by-n matrices over $\mathbb{R}$, 1
A	an n-by-n matrix, 1
$p_A(t)$	characteristic polynomial of A, 1
$\sigma(A)$	spectrum (set of eigenvalues) of A, 1
$\mathrm{m}_A(\lambda)$	algebraic multiplicity of λ as an eigenvalue of A, 1
$\mathrm{gm}_A(\lambda)$	geometric multiplicity of λ as an eigenvalue of A, 1
$\operatorname{rank} A$	rank of A, 1
I	identity matrix, 1
A^{T}	transpose of A, 2
A^*	Hermitian adjoint of A, 2
$A(J)$	principal submatrix of A resulting from deleting the rows and columns indexed by J, 2
$A[J]$	principal submatrix of A resulting from keeping the rows and columns indexed by J, 2
$A(j)$	principal submatrix of A resulting from deleting the row and column j, 2
$A[j]$	jth diagonal entry of A, 2
$\|J\|$	cardinality of the set J, 2
$\rho(A)$	spectral radius of A, 4
G	a graph, 5
$\mathcal{V}(G)$	vertex set of G, 5
$\mathcal{E}(G)$	edge set of G, 5
$\deg_G(v)$	degree of vertex v in G, 5

• Chapter 0	(cont.)
$G[\alpha]$	subgraph of G induced by the vertices α, 5
$G - v$	subgraph of G induced by all vertices of G, other than v, 5
$G - \alpha$, $G(\alpha)$	subgraph of G induced by all vertices of G not in α, 5
P_n	simple path on n vertices, 6
C_n	simple cycle on n vertices, 6
K_n	complete graph on n vertices, 6
$K_{m,n}$	complete bipartite graph, 6
T	a tree, 6
HDV	high-degree vertex, 7
$\mathrm{d}(T)$	diameter of T, 7
(a_{ij})	matrix in which the entry (i, j) is a_{ij}, 7
$\mathrm{G}(A)$	graph of A, 7
$\mathcal{H}(G)$	set of all Hermitian matrices whose graph is G, 7
$\mathcal{S}(G)$	set of all real symmetric matrices whose graph is G, 7
$A[H]$	principal submatrix of A resulting from keeping the rows and columns indexed by the vertices of a subgraph H, 7
$A(H)$	principal submatrix of A resulting from deleting the rows and columns indexed by the vertices of a subgraph H, 7
$T_1, \ldots, T_k$	k branches of T at v, 8
$\mathcal{I}$	partition, 9
$\mathcal{I}^*$	conjugate partition of $\mathcal{I}$, 9
$\preceq$	majorization of partitions, 9
• Chapter 1	
(1^n)	multiplicity list $(1, 1, \ldots, 1)$ with n 1s, 10
$\mathcal{L}(G)$	set of unordered multiplicity lists among matrices whose graph is G (the catalog for G), 11
$\mathcal{L}_o(G)$	set of ordered multiplicity lists among matrices whose graph is G (the ordered catalog for G), 11
$\mathrm{P}(T)$	path cover number of a tree T, 14
• Chapter 2	
$\oplus$	direct sum, 22
s-Parter	singly Parter vertex, 26
m-Parter	multiply Parter vertex, 26
P	“Parter” status of a vertex, 28
N	“neutral” status of a vertex, 28
D	“downer” status of a vertex, 28
PSD	positive semidefinite matrix, 48
PD	positive definite matrix, 48

• Chapter 3	
$\mathrm{M}(G)$	maximum multiplicity of an eigenvalue among matrices in $\mathcal{H}(G)$, 51
mr	minimum rank, 51
$\mathrm{mr}(G)$	minimum rank among matrices in $\mathcal{H}(G)$, 51
$\mathrm{P}(T)$	path cover number of T, 52
$\Delta(T)$	$\max[p-q]$ such that there exist q vertices of T whose removal from T leaves p paths, 53
RPM	residual path maximizing, 53
p_Q	the number of components of $G-Q$ (G graph and $Q \subset \mathcal{V}(G)$), 54
q_Q	the cardinality of Q (G graph and $Q \subset \mathcal{V}(G)$), 54
$H(T)$	the subgraph of T induced by the HDVs vertices in T, 55
$\delta(v)$	the number of neighbors of v that are not HDV, 55
$\|Q\|$	cardinality of the set Q, 55
$d_1, \ldots, d_n$	vertex degrees of a tree on n vertices, 65
$e(H)$	the number of edges present in $H(T)$, 65
• Chapter 4	
E_{ii}	square matrix with the ith diagonal entry 1 and with zeros elsewhere, 69
$\mathrm{m}_p(\lambda)$	multiplicity of λ as a root of the polynomial p, 69
$\mathbb{F}$	a general field, 70
$\mathbb{F}[x]$	set of polynomials with coefficients in $\mathbb{F}$, 70
$\mathrm{tr}\, A$	trace of matrix A, 73
$\|\alpha\|$	cardinality of the set α, 74
$\mathcal{F} = (\mathcal{P}, \mathcal{D}, \mathcal{N})$	fundamental decomposition, 80
FD	fundamental decomposition, 80
$\mathcal{P}$	set of all m-Parter vertices of a tree, 80
$\mathcal{D}$	set of downer regions of a tree, 80
$\mathcal{N}$	set of neutral regions of a tree, 80
$E_A(\lambda)$	eigenspace of A associated with λ, 82
$E'_{A(i)}(\lambda)$	set formed by extending every vector of $E_{A(i)}(\lambda)$ by a 0 in the ith coordinate, 83
$A(e_{ij})$	matrix $A = (a_{ij})$ altered only so that $a_{ij} = a_{ji} = 0$, 90
$G(e_{ij})$	the graph obtained from G by removal of the edge $\{i, j\}$, 90
• Chapter 5	
NIM	no intermediate multiplicities, 101
$\|Q\|$	cardinality of the set Q, 105

• Chapter 5	(cont.)
$\mathrm{M}_2(T)$	the largest sum of the top two multiplicities over lists in the catalog for T, 108
$\mathrm{D}_i(T)$	the number of degree i vertices in T, 108
• Chapter 6	
$\mathrm{c}(T)$	the minimum number of distinct eigenvalues among matrices in $\mathcal{S}(T)$, 110
$\mathrm{C}(d)$	$= \max\limits_{T:\mathrm{d}(T)=d} \mathrm{c}(T)$, 110
$\mathrm{d}(G)$	diameter of G, 110
CBD	combinatorial branch duplication, 112
ABD	algebraic branch duplication, 115
$\mathrm{C}(d) - d$	disparity for a given diameter d, 128
$\mathrm{U}(T)$	the fewest 1s in a list in $\mathcal{L}(T)$, 132
$\mathrm{U}(A)$	the number of eigenvalues with multiplicity 1 in the real symmetric matrix A, 132
$\mathrm{D}_2(T)$	the number of degree 2 vertices in T, 134
$k(m, T)$	the smallest nonnegative integer k such that there exist k distinct vertices of T whose removal from T leaves at least $m + k$ components, 136
$k'(m, T)$	the least k such that $1 + \sum_{i=1}^{k} (d_i - 1) \geq m + k$, in which $d_1 \geq \cdots \geq d_n$ is the degree sequence of T, 136
$\mathrm{l}_A(\lambda)$	the number of eigenvalues of A strictly to the left of a real number λ, 137
$\mathrm{r}_A(\lambda)$	the number of eigenvalues of A strictly to the right of a real number λ, 137
$\mathrm{b}_A(\alpha, \beta)$	the number of eigenvalues of A strictly between the real numbers α and β, 137
$\|Q_1\|$	cardinality of the set Q_1, 145
• Chapter 7	
IFT	implicit function theorem, 146
$\mathcal{A}$	an assignment of a tree, 148
$\mathcal{Z}(T)$	the collection of all subtrees of T, including T, rather than the power set of the vertices in T, 148
$\|V_i\|$	cardinality of the set V_i, 157
IEP	inverse eigenvalue problem, 161
GIEP	general inverse eigenvalue problem, 161

• CHAPTER 8	
g-star	generalized star, 167
l_i	length of an arm of a generalized star, 167
S_n	simple star on n vertices, 167
P_n	path on n vertices, 169
l^*	conjugate partition of l, 174
u_e	partition of N obtained from a partition u of M ($M \leq N$) appending $N - M$ 1's to the partition u, 174
$\hat{q} = \widehat{(q_1, \ldots, q_r)}$	an upward multiplicity list, 181
$\widehat{\mathcal{L}}_v(T)$	set of upward multiplicity lists at v among matrices in $\mathcal{S}(T)$ (the upward catalog for T at v), 182
• CHAPTER 9	
$D(T_1, T_2)$	double generalized star resulting from joining, by an edge, the central vertices of g-stars T_1 and T_2, 186
$q(A)$	the ordered multiplicity list of a real symmetric matrix A, 189
b^+	a list generated by the superposition principle from the upward multiplicity list $\hat{b}$, 191
• CHAPTER 10	
$\mathrm{L}(T_1, s_1, \ldots, T_k)$	a k-linear tree with components $T_1, \ldots, T_k$, 201
$\widehat{\mathcal{L}}_c(T)$	the collection of complete upward multiplicity lists for a g-star T, 202
LSP	linear superposition principle, 202
• CHAPTER 11	
K_n	the complete graph, 211
TPE	tree + an edge, 214
$\|Q\|$	cardinality of the set Q, 218
$K_{2,3}$	the complete bipartite graph on two and three vertices, 223
hK_4	graph that is homeomorph of K_4, 223
$hK_{2,3}$	graph that is homeomorph of $K_{2,3}$, 223
$\mathrm{M}(A)$	the maximum multiplicity of the eigenvalues of A, 223
$DM_{k,n}$	the set of connected graphs on n vertices that permit the multiplicity list $\{n - k, k\}$ for a positive integer k, $1 \leq k \leq \frac{n}{2}$, 226
DM-graph	dual multiplicity graph, 226
skeleton(G)	skeleton of a graph G, 228
$G_{1,1,\ldots,1}$	$K_n - k$ independent edges, 230

• CHAPTER 12

$\text{gm}_A(\lambda)$	geometric multiplicity of λ as an eigenvalue of A, 232
$\mathcal{M}_n(\mathbb{F})$	n-by-n matrices over the field $\mathbb{F}$, 232
$\mathcal{F}(G)$	set of all combinatorially symmetric matrices, over the field $\mathbb{F}$, whose graph is G, 232
$\|\alpha\|$	cardinality of the set α, 233
$\mathcal{R}(T)$	set of all combinatorially symmetric matrices, over the field $\mathbb{R}$, whose graph is T, 240
$\text{RS}(A)$	row space of a matrix A, 240
$\text{CS}(A)$	column space of a matrix A, 240
e_i	the ith basic unit vector with a 1 in position i and 0s elsewhere, 240
$\text{gM}(T)$	maximum geometric multiplicity of an eigenvalue among matrices in $\mathcal{F}(T)$, 243
$\text{mrF}(T)$	minimum rank among matrices in $\mathcal{F}(T)$, 243
$\text{mF}(T)$	maximum rank deficiency among matrices in $\mathcal{F}(T)$, 243
$r_A(t)$	minimal polynomial of a matrix A, 245