0

Background

To accomplish our purpose here, we need certain background from matrix theory, graph theory and a few other areas. For simplicity of use, we will list many of these facts and give references for the reader who would like to see proofs or know more. We confine ourselves here to just those ideas that we use, even though these topics are much broader.

0.1 Matrices

The n-by-n matrices with complex (resp. real) entries are denoted $\mathcal{M}_n(\mathbb{C})$ (resp. $\mathcal{M}_n(\mathbb{R})$). An ***eigenvalue*** λ (resp. ***eigenvector*** x) of $A \in \mathcal{M}_n(\mathbb{C})$ is a number $\lambda \in \mathbb{C}$ (resp. vector $0 \neq x \in \mathbb{C}^n$) such that

$$Ax = \lambda x$$

which is necessarily a root of the ***characteristic polynomial*** $p_A(t) = \det(tI - A)$. The set of all eigenvalues of A is denoted by $\sigma(A)$. For a given $\lambda \in \sigma(A)$, the (***algebraic***) ***multiplicity*** of λ is the number of times λ occurs as a root of $p_A(t)$, which we denote as

$$\mathrm{m}_A(\lambda).$$

If $A \in \mathcal{M}_n(\mathbb{C})$, there are always n eigenvalues, counting multiplicities, i.e., $\sum_{\lambda\in\sigma(A)} \mathrm{m}_A(\lambda) = n$. The ***geometric multiplicity*** of $\lambda \in \sigma(A)$, which we denote by $\mathrm{gm}_A(\lambda)$, is $n - \mathrm{rank}(A - \lambda I)$, and the geometric multiplicity is never more than the algebraic multiplicity.

Some good general references about matrices are [HJ13] and [HJ91], and a centrist elementary linear algebra book is [Lay]. We assume the content of a thorough elementary linear algebra course, such as may be given from [Lay], throughout.

0.1.1 Hermitian / Real Symmetric Matrices

The ***symmetric*** matrices in $\mathcal{M}_n(\mathbb{R})$ are those for which $A^T = A$ and, more generally, the ***Hermitian*** matrices in $\mathcal{M}_n(\mathbb{C})$ are those for which $A^* = A$. All Hermitian (resp. real symmetric) matrices have only real eigenvalues and are diagonalizable by ***unitary*** (resp. ***orthogonal***) matrices. This means that they may be written in the form $A = U^*DU$ in which $D \in \mathcal{M}_n(\mathbb{R})$ is diagonal and U has orthonormal columns (and is real in the symmetric case). For Hermitian matrices, the algebraic and geometric multiplicity of an eigenvalue are the same, which is very important throughout. Since the real symmetric matrices are included among the Hermitian matrices, everything that we say about the latter applies to the former.

0.1.2 Interlacing Eigenvalues

If a principal submatrix is extracted from an Hermitian matrix $A \in \mathcal{M}_n(\mathbb{C})$ by deleting the same row and column i, then the resulting matrix $B \in \mathcal{M}_{n-1}(\mathbb{C})$ is again Hermitian and has real eigenvalues. If the ordered eigenvalues of A are

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$$

and those of B are

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-1},$$

then

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \beta_{n-1} \leq \alpha_n,$$

the ***interlacing inequalities***.

In general we denote the principal submatrix of $A \in \mathcal{M}_n(\mathbb{C})$ resulting from deleting (resp. keeping) the rows and columns indexed by the subset $J \subseteq \{1, \ldots, n\}$ by $A(J)$ (resp. $A[J]$). In case $J = \{j\}$, we abbreviate $A(J)$ to $A(j)$. Of course $A[j]$ is the jth diagonal entry of A. In the prior paragraph, B is just $A(i)$. If more rows and (the same) columns are deleted, as in $A(J)$, then the interlacing inequalities may be applied multiple times to obtain inequalities such as

$$\alpha_i \leq \gamma_i \leq \alpha_{i+k}$$

if $|J| = k$, and $\gamma_1 \leq \cdots \leq \gamma_{n-k}$ are the eigenvalues of $A(J)$.

This and more about Hermitian matrices, eigenvalues and interlacing may be found in [HJ13]. A simple but important consequence of the interlacing inequalities is that

$$\left|\mathrm{m}_A(\lambda) - \mathrm{m}_{A(i)}(\lambda)\right| \leq 1$$

for Hermitian $A \in \mathcal{M}_n(\mathbb{C})$, any $\lambda \in \mathbb{R}$ and an index i, $1 \leq i \leq n$, i.e., deleting a row and (the same) column from an Hermitian matrix can increase or decrease the multiplicity of an eigenvalue by 1 or leave it the same. No other possibilities occur. This is so even when $\mathrm{m}_A(\lambda) = 0$, and, as we shall see, all three possibilities can occur.

0.1.3 Rank Inequalities and Change in Hermitian Multiplicities

Since algebraic and geometric multiplicity of an eigenvalue of an Hermitian matrix are the same, the rank of a change in an Hermitian matrix is important for understanding any change in the multiplicities. The fundamental inequality about ranks of sums is the following. Let A and B be m-by-n matrices over a field. Then

$$\operatorname{rank} A - \operatorname{rank} B \leq \operatorname{rank}(A + B) \leq \operatorname{rank} A + \operatorname{rank} B.$$

Thus, if A and B are Hermitian,

$$|\mathrm{m}_{A+B}(\lambda) - \mathrm{m}_A(\lambda)| \leq \operatorname{rank} B.$$

This means, in particular, that a change in a diagonal entry can change a multiplicity by no more than 1 (the same for any rank 1 perturbation) or that a change in an edge weight (off-diagonal entry), or the introduction of an edge, can change a multiplicity by no more than 2. We will see that the 2 can be improved under certain circumstances.

0.1.4 Eigenvector Structure When a Submatrix Has the Same Eigenvalue

Suppose Hermitian matrices A and $A(j)$ have the same eigenvalue λ. What can be said about the corresponding eigenvectors? This was studied in [JK]. A very special case of the general theory developed is the following. Let $A \in \mathcal{M}_n(\mathbb{C})$ be Hermitian and $\lambda \in \sigma(A)$. Then there is an eigenvector x of A, associated with λ with a 0 component in position j if and only if $\lambda \in \sigma(A(j))$. If x has such a 0 component, it is an easy calculation that $\lambda \in \sigma(A(j))$. The converse is more interesting and important. It follows that if $\lambda \in \sigma(A) \cap \sigma(A(j))$, then there is an eigenvector with jth component 0. If $\mathrm{m}_A(\lambda) = 1$ and $\lambda \in \sigma(A(j_1)), \sigma(A(j_2)), \ldots, \sigma(A(j_k))$, then there is an eigenvector of A

associated with λ in which each of the components $j_1, \ldots, j_k$ is 0 and $\lambda \in \sigma(A(\{j_1, \ldots, j_k\}))$.

0.1.5 Perron-Frobenius Theory of Nonnegative Matrices

As we shall see, when T is a tree, all possible multiplicity lists occur even when we require the entries to be nonnegative. Thus, certain elements of the theory of irreducible (because our graphs are connected and are often trees) nonnegative matrices will be useful. (In fact, the matrices may be taken to be ***primitive***, i.e., some power is entry-wise positive.) We list these here.

$\rho(A)$ denotes the spectral radius. Of course, there are many other aspects of the Perron-Frobenius theory that we need not mention.

Suppose that $A \in \mathcal{M}_n(\mathbb{R})$ is an irreducible (see Section 0.2.3), entry-wise nonnegative matrix. Then,

1. $\rho(A)$ is an eigenvalue of A;
2. $\rho(A)$ has algebraic multiplicity 1;
3. if B is a proper principal submatrix of A, then $\rho(B) < \rho(A)$; and
4. there is an entry-wise positive eigenvector of A associated with $\rho(A)$, and no other eigenvalue has an entry-wise nonnegative eigenvector.

0.1.6 Entries of Matrix Powers

For a positive integer k, by A^k, we mean $AA \cdots A$ (k-times). Just as the entries of AB (or A^2) may be written as sums of products of entries from A and B, the (p, q) entry of A^k is just a sum of products of entries from $A = (a_{ij})$. Let $1 \leq r_1, r_2, \ldots, r_{k-1} \leq n$ be any sequence of indices, repeats allowed, and let $a_{pr_1}a_{r_1r_2} \cdots a_{r_{k-1}q}$ be a k-fold p, q product. Then the (p, q) entry of A^k, $(A^k)_{pq}$, is the sum of all distinct k-fold p, q products:

$$(A^k)_{pq} = \sum_{r_1, r_2, \ldots, r_{k-1}} a_{pr_1}a_{r_1r_2} \cdots a_{r_{k-1}q}.$$

The sum is over all distinct (ordered) sequences of $k-1$ indices. Of course, this may be viewed in terms of directed path products in a weighted graph, and there is an analogous formula for distinct factors $(A_1A_2 \cdots A_k)_{pq}$. Many of the summands may be 0, and the sum is 0 if all summands are necessarily 0, which will occur if there is no k-fold path in the above mentioned graph.

0.1.7 M-matrices

We note that an n-by-n matrix is called an ***M-matrix*** (possibly singular) if it is of the form $\alpha I - A$, in which A is an n-by-n nonnegative matrix and $\alpha \geq \rho(A)$ (see [HJ91]).

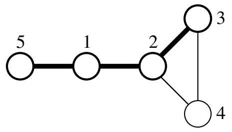
Cambridge University Press
978-1-107-09545-8 — Eigenvalues, Multiplicities and Graphs
Charles R. Johnson , Carlos M. Saiago
Excerpt
More Information

We next mention some elementary ideas about graphs and set some of our notation.

0.2 Graphs

0.2.1 Definitions

A simple, undirected ***graph*** consists of a set of ***vertices*** and a set of ***edges*** (2-membered subsets of the vertices) without "loops" or repeated edges. We will just use the word "graph" throughout. It is convenient to think of a graph pictorially. For example,

$G =$ 

has vertices 1, 2, 3, 4, 5 and edges $\{1, 2\}$, $\{1, 5\}$, $\{2, 3\}$, $\{2, 4\}$ and $\{3, 4\}$. Note that for most purposes, the actual labeling of the vertices is unnecessary and for referential convenience only. $\mathcal{V}(G)$ denotes the vertex set of G and the ***degree*** of a vertex v, denoted $\deg_G(v)$, is the number of edges to which v belongs. The edge set of G is denoted by $\mathcal{E}(G)$.

A ***path*** in a graph is an ordered list of edges of the graph:

$$\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{k-1}, i_k\}.$$

This path is from i_1 to i_k. For example, the highlighted edges in the graph above constitute a path from 3 to 5 (via 2 and 1). A graph is said to be ***connected*** if, for every pair of distinct vertices, there is a path from one to the other. A path is ***simple*** if no vertex appears in more than two (consecutive) edges, and a ***cycle*** is a path in which each vertex appears in exactly two edges: a simple path from a vertex to itself.

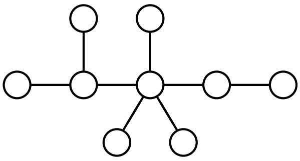
A ***subgraph*** of a given graph G is another graph H, each of whose vertex and edge sets is a subset of that of G. A ***supergraph*** is another graph of which G is a subgraph. The subgraph H is ***induced*** if it contains all edges of G among the vertices of H. For example, if α is a subset of the vertices of G, $G[\alpha]$ is the subgraph induced by the vertices α. If v is a vertex of G, we use $G - v$ to denote the subgraph induced by all vertices, other than v. If α is a subset of the vertices of G, we use $G - \alpha$ or $G(\alpha)$ to denote the subgraph of G induced by all vertices of G not in α. Note the similarity to principal submatrices of a square matrix.

New graphs may be obtained by simple changes in other graphs in a variety of ways. Examples include: (1) adding an edge where there was none before (without changing the number of vertices); (2) adding a new edge and a vertex pendent at an existing vertex; or (3) ***edge subdivision***, in which a new vertex of degree 2 is positioned along an existing edge.

Several special graphs that can occur on any number of vertices are accorded particular notation: P_n is the graph consisting of a single, simple path on n vertices (“***the path***”); C_n is the single, simple cycle on n vertices (“***the cycle***”); and K_n is the ***complete graph*** on n vertices (all possible edges). A graph is ***bipartite*** if its vertices may be partitioned into two disjoint subsets such that all edges connect vertices in one subset to vertices in the other. In $K_{m,n}$, the ***complete bipartite graph***, on m and n vertices, the two subsets have m and n vertices, respectively, and all possible (mn) edges occur between them.

0.2.2 Trees

A ***tree*** is simply a minimally connected, undirected graph T, i.e., a connected, undirected graph on n vertices with just $n-1$ edges. For example,

$T =$ 

is a tree with 9 vertices and 8 edges. Trees are very important among all graphs; for example, a ***spanning tree*** of a connected graph is a subgraph with the same set of vertices that is a tree. And trees have very special structure among graphs. In several ways, the subject of multiplicities has added to the understanding of this structure. And certainly, the subject of multiplicities is most structured when the underlying graph is a tree, as we will see.

Since a tree is connected, there is a path between i and j for each pair of distinct vertices i and j. Moreover, there is only one simple path (which characterizes trees) and this one has the minimum number of vertices among all paths. A vertex of a tree may have any degree, but the sum of all degrees is fixed at $2(n-1)$ when there are n vertices. Two different trees may have the same set of degrees, but there are simple conditions that a partition of $2(n-1)$ into n positive parts be a degree sequence for a tree [ChaLes]. Given these conditions, it is easy to construct some trees with the given degree sequence. Of course, there must be at least two vertices of degree 1 in a nontrivial tree, and these are called ***pendent***. There are only two exactly when the tree is a path. Vertices of degree 2 are also of special importance, but these may or may not occur. We refer to vertices of all other degrees (at least 3) as ***high-degree vertices*** (HDVs); each non-path tree has at least one, and they also play a special role for us. A tree is called ***linear*** if all its HDVs lie along a simple induced path of the tree. A ***star*** is just a tree on n vertices having a vertex of degree $n-1$. If $n \geq 3$, then the degree $n-1$ vertex is called the ***central vertex*** of the star. An induced path of a tree with the greatest number of vertices is called a ***diameter*** (and this number of vertices is the diameter $\mathrm{d}(T)$ of the tree). All trees are bipartite graphs. (Note that in some literature, diameter is measured by the number of edges in such a path, but here, our measure is more convenient. The difference is exactly 1.)

A ***forest*** is simply a collection of trees, i.e., a graph on n vertices with no more than $n-1$ edges and no cycles. Induced subgraphs of trees (e.g., resulting from the removal of a vertex) are forests. A tree is called ***binary*** if it has vertices only of degree ≤ 3 and ***complete*** (or ***full***) ***binary*** if it is binary with no vertex of degree 2.

0.2.3 Graphs and Matrices

One important use of graphs is as an accounting device for the nonzero entries of a matrix, and this can contribute important insights expositorially. Given an Hermitian matrix $A \in \mathcal{M}_n(\mathbb{C})$, the graph of $A = (a_{ij})$, $\mathrm{G}(A)$, is the graph on n vertices $1, \ldots, n$ with an edge $\{i, j\}$ if and only if $i \neq j$ and $a_{ij} \neq 0$. So, $\mathrm{G}(A)$ just identifies the positions of the nonzero (and thus zero) off-diagonal entries of A. For us, it is crucial to think of all Hermitian or real symmetric matrices with the same graph G. We let $\mathcal{H}(G)$ denote the set of all Hermitian matrices whose graph is G, and $\mathcal{S}(G)$ the set of all real symmetric matrices with graph G. We emphasize that G presents no constraints on the diagonal entries of the matrices in $\mathcal{H}(G)$ or $\mathcal{S}(G)$, except, of course, that they are real.

Sometimes an index set $J \subseteq \{1, \ldots, n\}$ will be indicated indirectly. For example, if H is a subgraph of G, then $A[H]$ (resp. $A(H)$) is $A[J]$ (resp. $A(J)$), in which J is the set of indices corresponding to the vertices of H.

A square matrix A is said to be ***reducible*** if there is a permutation matrix P such that

$$P^{-1}AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

in which B and D are square matrices and 0 is a zero matrix. Matrix A is called ***irreducible*** if it is not reducible. If A is symmetric and reducible, then we have $C = 0$ and, therefore, the graph of A is disconnected. Thus, the graph of a symmetric matrix A is connected if and only if A is irreducible. The eigenvalues of a reducible matrix are just the union (counting multiplicities) of the eigenvalues of its diagonal blocks. Thus, in the study of multiplicities, it suffices to consider only connected graphs (the graphs of irreducible components).

0.2.4 Graphs and Characteristic Polynomial Formulae

When T is a tree and v is a vertex of T, for each $A \in \mathcal{H}(T)$ the matrix $A(v)$ is a direct sum whose summands correspond to components of $T - v$, which we call ***branches*** of T at v. If $\deg_T(v) = k$, usually we denote by $T_1, \ldots, T_k$ the k branches of T at v and by u_i the neighbor of v in the branch T_i.

We shall use expansions of the characteristic polynomial $p_A(t)$ of an Hermitian matrix $A = \left(a_{ij}\right)$, whose graph is a tree T.

A useful one, which we call the ***neighbors formula***, is obtained when attention is focused upon the edges connecting a particular vertex v to its neighbors $u_1, \ldots, u_k$ in T. We have

$$p_A(t) = (t - a_{vv}) \prod_{j=1}^{k} p_{A[T_j]}(t) - \sum_{j=1}^{k} \left|a_{vu_j}\right|^2 p_{A[T_j - u_j]}(t) \prod_{\substack{l=1 \\ l \neq j}}^{k} p_{A[T_l]}(t), \qquad (1)$$

in which T_j is the branch of T at v, containing u_j. This notation will be used throughout.

We call another useful expansion of the characteristic polynomial the ***bridge formula***. It is obtained when attention is focused upon the edge connecting two vertices v and u_j. Denoting by T_v the component of T resulting from deletion of u_j and containing v, we have

$$p_A(t) = p_{A[T_v]}(t) p_{A[T_j]}(t) - \left|a_{vu_j}\right|^2 p_{A[T_v - v]}(t) p_{A[T_j - u_j]}(t). \qquad (2)$$

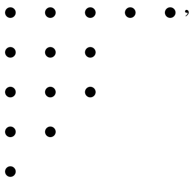
(In (1) and (2), we observe the standard convention that the characteristic polynomial of the empty matrix is identically 1.)

Both expansions appear in [P], and in [MOleVWie], a detailed account of several expansions of the characteristic polynomial is presented in graph-theoretical language.

0.3 Other Background

A ***partition*** of a positive integer n into $k \leq n$ ***parts*** is a list $\mathcal{I} = (i_1, \ldots, i_k)$ of positive integers $i_1 \geq \cdots \geq i_k$ whose sum is n. When k is not specified, we just say “partition.” Since an n-by-n Hermitian matrix has exactly n eigenvalues, counting multiplicities, a multiplicity list for an n-by-n matrix forms just a certain (possible unordered) partition of n. So in a sense, we study partitions (of n).

One way to describe a partition is with a rectilinear dot diagram. For example, the partition of 14 into parts 5, 3, 3, 2, 1 may be viewed as



in which the number of dots in a row is a part of the partition. Given such a diagram, the ***conjugate partition*** may be described. It is just the partition corresponding to the transpose of the diagram, or the one whose parts are the numbers of dots per column. The conjugate partition may also be described by counting the number of parts (at least 1), then the number of parts at least 2, and so on, then taking first differences. Thus, if $\mathcal{I} = (i_1, \ldots, i_k)$ is a partition with $i_1 \geq \cdots \geq i_k$, the conjugate partition of $\mathcal{I}$ is then $\mathcal{I}^* = (i_1^*, \ldots, i_{i_1}^*)$, in which i_r^* is the number of js such that $i_j \geq r$. Note that $i_1^* \geq \cdots \geq i_{i_1}^* \geq 1$ and that $\mathcal{I}$ and $\mathcal{I}^*$ are partitions of the same integer.

Given two partitions of n, say $\mathcal{I} = (i_1, i_2, \ldots, i_k)$ and $\mathcal{J} = (j_1, j_2, \ldots, j_s)$, the partition $\mathcal{I}$ is said to be ***majorized*** by $\mathcal{J}$, and we write $\mathcal{I} \preceq \mathcal{J}$, if the inequalities

$$\begin{gathered} i_1 \leq j_1 \\ i_1 + i_2 \leq j_1 + j_2 \\ i_1 + i_2 + i_3 \leq j_1 + j_2 + j_3 \\ \vdots \\ i_1 + \cdots + i_k \leq j_1 + \cdots + j_s \end{gathered}$$

are satisfied. Of course, this necessitates that $s \leq k$. Since both $\mathcal{I}$ and $\mathcal{J}$ are partitions of n, the last inequality in such a list is necessarily an equality, and this is part of the definition of ***majorization***. When the last inequality is not required to be equality (as it is above), the concept is called ***weak majorization***.

Majorization arises in a remarkable number of ways in mathematics and plays a major role in matrices and inequalities [MarOlk]. The way it arises in this subject (Chapter 8) is a new one.

1

Introduction

1.1 Problem Definition

For any graph G, there will be (many) matrices in $\mathcal{S}(G)$, or $\mathcal{H}(G)$, with distinct eigenvalues, and these eigenvalues can be any (distinct) real numbers. But for some time, it has been realized that the graph of an Hermitian matrix can substantially constrain the possible multiplicities of its eigenvalues. For example, an irreducible tridiagonal Hermitian matrix must have distinct eigenvalues; this is the case in which G is a path (see Section 2.7). Not surprisingly, if G is a tree and contains a relatively long path, it must have many different eigenvalues (see Section 6.2), but if it is not a path, it does allow some multiple eigenvalues.

What lists of multiplicities for the eigenvalues may then occur among the Hermitian or real symmetric matrices with a given graph G on n vertices? Apparently, the list $(1, 1, \ldots, 1)$, which we abbreviate to (1^n), does occur, and for all nonpaths, other lists occur as well in $\mathcal{H}(G)$ or $\mathcal{S}(G)$. First, we formalize the question.

Given a list of eigenvalues, including multiplicities, that occurs for an n-by-n Hermitian or real symmetric matrix, the multiplicities may be summarized in two ways: (1) a simple partition of n in which the parts are the multiplicities of the distinct eigenvalues, usually listed in descending order; and (2) the ordered version of (1) in which the order of the parts respects the numerical order of the underlying (real number) eigenvalues. We refer to the former as an ***unordered multiplicity list*** and the latter as an ***ordered multiplicity list***. We may also use exponents to indicate the frequency of a multiplicity.

For example, if $n = 14$ and the eigenvalues are

$$-3, -1, -1, 2, 4, 4, 4, 5, 5, 6, 8, 8, 10, 11,$$

then, as an unordered multiplicity list, this would become

$$(3, 2^3, 1^5), \qquad \text{or} \qquad (3, 2, 2, 2, 1, 1, 1, 1, 1),$$