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978-1-107-09234-1 - Sobolev Spaces on Metric Measure Spaces: An Approach Based on Upper Gradients

Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam and Jeremy T. Tyson

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