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978-1-107-09234-1 - Sobolev Spaces on Metric Measure Spaces: An Approach Based on Upper Gradients

Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam and Jeremy T. Tyson  
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## Sobolev Spaces on Metric Measure Spaces

Analysis on metric spaces emerged in the 1990s as an independent research field providing a unified treatment of first-order analysis in diverse and potentially nonsmooth settings. Based on the fundamental concept of the upper gradient, the notion of a Sobolev function was formulated in the setting of metric measure spaces supporting a Poincaré inequality.

This coherent treatment from first principles is an ideal introduction to the subject for graduate students and a useful reference for experts. It presents the foundations of the theory of such first-order Sobolev spaces, then explores the geometric implications of the critical Poincaré inequality and indicates numerous examples of spaces satisfying this axiom. A distinguishing feature of the book is its focus on vector-valued Sobolev spaces. The final chapters include proofs of several landmark theorems, including Cheeger's stability theorem for Poincaré inequalities under Gromov–Hausdorff convergence and the Keith–Zhong self-improvement theorem for Poincaré inequalities.

JUHA HEINONEN (1960–2007) was Professor of Mathematics at the University of Michigan. His principal areas of research interest included quasiconformal mappings, nonlinear potential theory, and analysis on metric spaces. He was the author of over 60 research articles, including several published posthumously, and two textbooks. A member of the Finnish Academy of Science and Letters, Heinonen received the Excellence in Research Award from the University of Michigan in 1997 and gave an invited lecture at the International Congress of Mathematicians in Beijing in 2002.

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# Sobolev Spaces on Metric Measure Spaces

## An Approach Based on Upper Gradients

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Frontmatter[More information](#)


---

## Contents

---

	<i>Preface</i>	<i>page xi</i>
<b>1</b>	<b>Introduction</b>	1
<b>2</b>	<b>Review of basic functional analysis</b>	7
	2.1 Normed and seminormed spaces	7
	2.2 Linear operators and dual spaces	13
	2.3 Convergence theorems	16
	2.4 Reflexive spaces	24
	2.5 Notes to Chapter 2	34
<b>3</b>	<b>Lebesgue theory of Banach space-valued functions</b>	36
	3.1 Measurability for Banach space-valued functions	36
	3.2 Integrable functions and the spaces $L^p(X : V)$	42
	3.3 Metric measure spaces	49
	3.4 Differentiation	73
	3.5 Maximal functions	90
	3.6 Notes to Chapter 3	96
<b>4</b>	<b>Lipschitz functions and embeddings</b>	98
	4.1 Lipschitz functions, extensions, and embeddings	98
	4.2 Lower semicontinuous functions	107
	4.3 Hausdorff measures	110
	4.4 Functions with bounded variation	111
	4.5 Notes to Chapter 4	119
<b>5</b>	<b>Path integrals and modulus</b>	121
	5.1 Curves in metric spaces	121
	5.2 Modulus of a curve family	127
	5.3 Estimates for modulus	134
	5.4 Notes to Chapter 5	141

Cambridge University Press

978-1-107-09234-1 - Sobolev Spaces on Metric Measure Spaces: An Approach Based on Upper Gradients

Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam and Jeremy T. Tyson  
Frontmatter[More information](#)

viii

*Contents*

<b>6</b>	<b>Upper gradients</b>	143
6.1	Classical first-order Sobolev spaces	143
6.2	Upper gradients	151
6.3	Maps with $p$ -integrable upper gradients	156
6.4	Notes to Chapter 6	166
<b>7</b>	<b>Sobolev spaces</b>	167
7.1	Vector-valued Sobolev functions on metric spaces	167
7.2	The Sobolev $p$ -capacity	183
7.3	$N^{1,p}(X : V)$ is a Banach space	190
7.4	The space $HN^{1,p}(X : V)$ and quasicontinuity	198
7.5	Main equivalence classes and the $MEC_p$ property	201
7.6	Notes to Chapter 7	203
<b>8</b>	<b>Poincaré inequalities</b>	205
8.1	Poincaré inequality and pointwise inequalities	205
8.2	Density of Lipschitz functions	228
8.3	Quasiconvexity and the Poincaré inequality	233
8.4	Continuous upper gradients and pointwise Lipschitz constants	238
8.5	Notes to Chapter 8	243
<b>9</b>	<b>Consequences of Poincaré inequalities</b>	245
9.1	Sobolev–Poincaré inequalities	245
9.2	Lebesgue points of Sobolev functions	261
9.3	Measurability of equivalence classes and $MEC_p$	271
9.4	Annular quasiconvexity	279
9.5	Notes to Chapter 9	282
<b>10</b>	<b>Other definitions of Sobolev-type spaces</b>	285
10.1	The Cheeger–Sobolev space	285
10.2	The Hajlasz–Sobolev space	286
10.3	Sobolev spaces defined via Poincaré inequalities	290
10.4	The Korevaar–Schoen–Sobolev space	294
10.5	Summary	304
10.6	Notes to Chapter 10	304
<b>11</b>	<b>Gromov–Hausdorff convergence and Poincaré inequalities</b>	306
11.1	The Gromov–Hausdorff distance	306
11.2	Gromov’s compactness theorem	312
11.3	Pointed Gromov–Hausdorff convergence	315
11.4	Pointed measured Gromov–Hausdorff convergence	324



Cambridge University Press

978-1-107-09234-1 - Sobolev Spaces on Metric Measure Spaces: An Approach Based on Upper Gradients

Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam and Jeremy T. Tyson  
Frontmatter[More information](#)

<i>Contents</i>		ix
11.5	Persistence of doubling measures under Gromov–Hausdorff convergence	327
11.6	Persistence of Poincaré inequalities under Gromov–Hausdorff convergence	330
11.7	Notes to Chapter 11	335
<b>12</b>	<b>Self-improvement of Poincaré inequalities</b>	<b>337</b>
12.1	Geometric properties of geodesic doubling metric measure spaces	337
12.2	Preliminary local arguments	340
12.3	Self-improvement of the Poincaré inequality	355
12.4	Notes to Chapter 12	363
<b>13</b>	<b>An introduction to Cheeger’s differentiation theory</b>	<b>364</b>
13.1	Asymptotic generalized linearity	364
13.2	Caccioppoli-type estimates	369
13.3	Minimal weak upper gradients of distance functions are nontrivial	371
13.4	The differential structure	373
13.5	Comparisons between $\rho_u$ and $\text{Lip } u$ , Taylor’s theorem, and the reflexivity of $N^{1,p}(X)$	377
13.6	Notes to Chapter 13	385
<b>14</b>	<b>Examples, applications, and further research directions</b>	<b>387</b>
14.1	Quasiconformal and quasisymmetric mappings	387
14.2	Spaces supporting a Poincaré inequality	392
14.3	Applications and further research directions	407
	<i>References</i>	412
	<i>Notation index</i>	427
	<i>Subject index</i>	429

Cambridge University Press

978-1-107-09234-1 - Sobolev Spaces on Metric Measure Spaces: An Approach Based  
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Frontmatter

[More information](#)

---

## Preface

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The aim of this book is to present a coherent and essentially self-contained treatment of the theory of first-order Sobolev spaces on metric measure spaces, based on the notion of *upper gradients*.

The project of writing this book was initiated by Juha Heinonen in 2000. His premature passing in 2007 significantly delayed progress in its preparation. We wish to thank Karen E. Smith for securing for us valuable private material of Juha Heinonen pertaining to this text.

Over the years of preparation of the manuscript, we have benefitted from discussions with, and advice from, many colleagues. Amongst them, we wish to give special thanks to the following individuals. We thank Luigi Ambrosio, Piotr Hajłasz, Ilkka Holopainen, Riikka Korte, Jan Malý, Anton Petrunin, and Stephen Semmes for valuable contributions to the mathematical content of this book. Bruce Hanson and Pietro Poggi-Corradini provided detailed comments and corrections of various drafts of the manuscript. We also acknowledge Sita Benedict, Anders Björn, Jana Björn, Estibalitz Durand Cartagena, Nicola Gigli, Changyu Guo, Nijjwal Karak, Aapo Kauranen, Panu Lahti, Marcos Lopez, Marie Snipes, and Thomas Zürcher for reading the manuscript and providing useful feedback. The Mathematica code used to create Figures 14.3 and 14.4 was written by Anton Lukyanenko.

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Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam and Jeremy T. Tyson  
Frontmatter

[More information](#)

---

xii

*Preface*

of the manuscript was completed during a snowy weekend in January 2014 at the Clifton Gaslight Bed and Breakfast in Cincinnati, Ohio. We wish to thank Scott and Maria Crawford for their hospitality.

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