Sobolev Spaces on Metric Measure Spaces

Analysis on metric spaces emerged in the 1990s as an independent research field providing a unified treatment of first-order analysis in diverse and potentially nonsmooth settings. Based on the fundamental concept of the upper gradient, the notion of a Sobolev function was formulated in the setting of metric measure spaces supporting a Poincaré inequality.

This coherent treatment from first principles is an ideal introduction to the subject for graduate students and a useful reference for experts. It presents the foundations of the theory of such first-order Sobolev spaces, then explores the geometric implications of the critical Poincaré inequality and indicates numerous examples of spaces satisfying this axiom. A distinguishing feature of the book is its focus on vector-valued Sobolev spaces. The final chapters include proofs of several landmark theorems, including Cheeger's stability theorem for Poincaré inequalities under Gromov–Hausdorff convergence and the Keith–Zhong self-improvement theorem for Poincaré inequalities.

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NEW MATHEMATICAL MONOGRAPHS

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An Approach Based on Upper Gradients

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University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107092341

First published 2015

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Sobolev spaces on metric measure spaces : an approach based on upper gradients / Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, Jeremy T. Tyson.
pages cm. – (New mathematical monographs)
Includes bibliographical references and indexes.
ISBN 978-1-107-09234-1 (Hardback)
1. Metric spaces. 2. Sobolev spaces. I. Heinonen, Juha. II. Koskela, Pekka. III. Shanmugalingam, Nageswari. IV. Tyson, Jeremy T., 1972–
QA611.28.S63 2015
515′.7–dc23 2014027794

ISBN 978-1-107-09234-1 Hardback

In memory of Frederick W. Gehring
(1925–2012)

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Preface

The aim of this book is to present a coherent and essentially self-contained treatment of the theory of first-order Sobolev spaces on metric measure spaces, based on the notion of *upper gradients*.

The project of writing this book was initiated by Juha Heinonen in 2000. His premature passing in 2007 significantly delayed progress in its preparation. We wish to thank Karen E. Smith for securing for us valuable private material of Juha Heinonen pertaining to this text.

Over the years of preparation of the manuscript, we have benefitted from discussions with, and advice from, many colleagues. Amongst them, we wish to give special thanks to the following individuals. We thank Luigi Ambrosio, Piotr Hajłasz, Ilkka Holopainen, Riikka Korte, Jan Malý, Anton Petrunin, and Stephen Semmes for valuable contributions to the mathematical content of this book. Bruce Hanson and Pietro Poggi-Corradini provided detailed comments and corrections of various drafts of the manuscript. We also acknowledge Sita Benedict, Anders Björn, Jana Björn, Estibalitz Durand Cartagena, Nicola Gigli, Changyu Guo, Nijjwal Karak, Aapo Kauranen, Panu Lahti, Marcos Lopez, Marie Snipes, and Thomas Zürcher for reading the manuscript and providing useful feedback. The Mathematica code used to create Figures 14.3 and 14.4 was written by Anton Lukyanenko.

The authors are grateful to Karen E. Smith and Kai Rajala for their encouragement in completing this project.

Our contributions to the field of research summarized in this book have been supported by grants from the US National Science Foundation, the Simons Foundation, and the Academy of Finland. We wish to acknowledge these agencies for their support over the years. Also, we would like to thank the Institute for Pure and Applied Mathematics and the University of Jyväskylä for their kind hospitality during some of the intensive writing periods. The final editing

of the manuscript was completed during a snowy weekend in January 2014 at the Clifton Gaslight Bed and Breakfast in Cincinnati, Ohio. We wish to thank Scott and Maria Crawford for their hospitality.

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