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Excerpt
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1

Poisson and Other Discrete Distributions

The Poisson distribution arises as a limit of the binomial distribution. This chapter contains a brief discussion of some of its fundamental properties as well as the Poisson limit theorem for null arrays of integer-valued random variables. The chapter also discusses the binomial and negative binomial distributions.

1.1 The Poisson Distribution

A random variable X is said to have a *binomial distribution* $\mathrm{Bi}(n, p)$ with parameters $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $p \in [0, 1]$ if

$$\mathbb{P}(X = k) = \mathrm{Bi}(n, p; k) := \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, \dots, n, \tag{1.1}$$

where $0^0 := 1$. In the case $n = 1$ this is the *Bernoulli distribution* with parameter p. If $X_1, \dots, X_n$ are independent random variables with such a Bernoulli distribution, then their sum has a binomial distribution, that is

$$X_1 + \cdots + X_n \overset{d}{=} X, \tag{1.2}$$

where X has the distribution $\mathrm{Bi}(n, p)$ and where $\overset{d}{=}$ denotes equality in distribution. It follows that the expectation and variance of X are given by

$$\mathbb{E}[X] = np, \qquad \mathbb{V}\mathrm{ar}[X] = np(1 - p). \tag{1.3}$$

A random variable X is said to have a *Poisson distribution* $\mathrm{Po}(\gamma)$ with parameter $\gamma \geq 0$ if

$$\mathbb{P}(X = k) = \mathrm{Po}(\gamma; k) := \frac{\gamma^k}{k!} e^{-\gamma}, \quad k \in \mathbb{N}_0. \tag{1.4}$$

If $\gamma = 0$, then $\mathbb{P}(X = 0) = 1$, since we take $0^0 = 1$. Also we allow $\gamma = \infty$; in this case we put $\mathbb{P}(X = \infty) = 1$ so $\mathrm{Po}(\infty; k) = 0$ for $k \in \mathbb{N}_0$.

The Poisson distribution arises as a limit of binomial distributions as follows. Let $p_n \in [0,1]$, $n \in \mathbb{N}$, be a sequence satisfying $np_n \to \gamma$ as $n \to \infty$, with $\gamma \in (0,\infty)$. Then, for $k \in \{0,\dots,n\}$,

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{(np_n)^k}{k!} \cdot \frac{(n)_k}{n^k} \cdot (1-p_n)^{-k} \cdot \left(1 - \frac{np_n}{n}\right)^n \to \frac{\gamma^k}{k!} e^{-\gamma}, \quad (1.5)$$

as $n \to \infty$, where

$$(n)_k := n(n-1)\cdots(n-k+1) \quad (1.6)$$

is the k-th *descending factorial* (of n) with $(n)_0$ interpreted as 1.

Suppose X is a Poisson random variable with finite parameter γ. Then its expectation is given by

$$\mathbb{E}[X] = e^{-\gamma} \sum_{k=0}^{\infty} k \frac{\gamma^k}{k!} = e^{-\gamma} \gamma \sum_{k=1}^{\infty} \frac{\gamma^{k-1}}{(k-1)!} = \gamma. \quad (1.7)$$

The *probability generating function* of X (or of $\text{Po}(\gamma)$) is given by

$$\mathbb{E}[s^X] = e^{-\gamma} \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} s^k = e^{-\gamma} \sum_{k=0}^{\infty} \frac{(\gamma s)^k}{k!} = e^{\gamma(s-1)}, \quad s \in [0,1]. \quad (1.8)$$

It follows that the *Laplace transform* of X (or of $\text{Po}(\gamma)$) is given by

$$\mathbb{E}[e^{-tX}] = \exp[-\gamma(1-e^{-t})], \quad t \geq 0. \quad (1.9)$$

Formula (1.8) is valid for each $s \in \mathbb{R}$ and (1.9) is valid for each $t \in \mathbb{R}$. A calculation similar to (1.8) shows that the *factorial moments* of X are given by

$$\mathbb{E}[(X)_k] = \gamma^k, \quad k \in \mathbb{N}_0, \quad (1.10)$$

where $(0)_0 := 1$ and $(0)_k := 0$ for $k \geq 1$. Equation (1.10) implies that

$$\mathbb{V}\text{ar}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[(X)_2] + \mathbb{E}[X] - \mathbb{E}[X]^2 = \gamma. \quad (1.11)$$

We continue with a characterisation of the Poisson distribution.

Proposition 1.1 *An $\mathbb{N}_0$-valued random variable X has distribution* $\text{Po}(\gamma)$ *if and only if, for every function $f : \mathbb{N}_0 \to \mathbb{R}_+$, we have*

$$\mathbb{E}[Xf(X)] = \gamma \mathbb{E}[f(X+1)]. \quad (1.12)$$

Proof By a similar calculation to (1.7) and (1.8) we obtain for any function $f : \mathbb{N}_0 \to \mathbb{R}_+$ that (1.12) holds. Conversely, if (1.12) holds for all such functions f, then we can make the particular choice $f := \mathbf{1}_{\{k\}}$ for $k \in \mathbb{N}$, to obtain the recursion

$$k\,\mathbb{P}(X = k) = \gamma\,\mathbb{P}(X = k-1).$$

This recursion has (1.4) as its only (probability) solution, so the result follows. □

1.2 Relationships Between Poisson and Binomial Distributions

The next result says that if X and Y are independent Poisson random variables, then $X+Y$ is also Poisson and the conditional distribution of X given $X+Y$ is binomial:

Proposition 1.2 *Let X and Y be independent with distributions* $\mathrm{Po}(\gamma)$ *and* $\mathrm{Po}(\delta)$*, respectively, with* $0 < \gamma+\delta < \infty$. *Then $X+Y$ has distribution* $\mathrm{Po}(\gamma+\delta)$ *and*

$$\mathbb{P}(X = k \mid X + Y = n) = \mathrm{Bi}(n, \gamma/(\gamma+\delta); k), \qquad n \in \mathbb{N}_0,\ k = 0, \dots, n.$$

Proof For $n \in \mathbb{N}_0$ and $k \in \{0, \dots, n\}$,

$$\begin{aligned}\mathbb{P}(X = k, X + Y = n) &= \mathbb{P}(X = k, Y = n - k) = \frac{\gamma^k}{k!}e^{-\gamma}\frac{\delta^{n-k}}{(n-k)!}e^{-\delta} \\ &= e^{-(\gamma+\delta)}\left(\frac{(\gamma+\delta)^n}{n!}\right)\binom{n}{k}\left(\frac{\gamma}{\gamma+\delta}\right)^k\left(\frac{\delta}{\gamma+\delta}\right)^{n-k} \\ &= \mathrm{Po}(\gamma+\delta; n)\,\mathrm{Bi}(n, \gamma/(\gamma+\delta); k),\end{aligned}$$

and the assertions follow. □

Let Z be an $\mathbb{N}_0$-valued random variable and let $Z_1, Z_2, \dots$ be a sequence of independent random variables that have a Bernoulli distribution with parameter $p \in [0, 1]$. If Z and $(Z_n)_{n\geq 1}$ are independent, then the random variable

$$X := \sum_{j=1}^{Z} Z_j \tag{1.13}$$

is called a *p-thinning* of Z, where we set $X := 0$ if $Z = 0$. This means that the conditional distribution of X given $Z = n$ is binomial with parameters n and p.

The following partial converse of Proposition 1.2 is a noteworthy property of the Poisson distribution.

Proposition 1.3 *Let* $p \in [0, 1]$. *Let Z have a Poisson distribution with parameter $\gamma \geq 0$ and let X be a p-thinning of Z. Then X and $Z - X$ are independent and Poisson distributed with parameters $p\gamma$ and* $(1 - p)\gamma$, *respectively.*

Proof We may assume that $\gamma > 0$. The result follows once we have shown that

$$\mathbb{P}(X = m, Z - X = n) = \mathrm{Po}(p\gamma; m)\,\mathrm{Po}((1-p)\gamma; n), \quad m, n \in \mathbb{N}_0. \tag{1.14}$$

Since the conditional distribution of X given $Z = m + n$ is binomial with parameters $m + n$ and p, we have

$$\begin{aligned}\mathbb{P}(X = m, Z - X = n) &= \mathbb{P}(Z = m+n)\,\mathbb{P}(X = m \mid Z = m+n)\\ &= \left(\frac{e^{-\gamma}\gamma^{m+n}}{(m+n)!}\right)\binom{m+n}{m} p^m (1-p)^n\\ &= \left(\frac{p^m\gamma^m}{m!}\right)e^{-p\gamma}\left(\frac{(1-p)^n\gamma^n}{n!}\right)e^{-(1-p)\gamma},\end{aligned}$$

and (1.14) follows. □

1.3 The Poisson Limit Theorem

The next result generalises (1.5) to sums of Bernoulli variables with unequal parameters, among other things.

Proposition 1.4 *Suppose for $n \in \mathbb{N}$ that $m_n \in \mathbb{N}$ and $X_{n,1}, \dots, X_{n,m_n}$ are independent random variables taking values in $\mathbb{N}_0$. Let $p_{n,i} := \mathbb{P}(X_{n,i} \geq 1)$ and assume that*

$$\lim_{n\to\infty} \max_{1\leq i\leq m_n} p_{n,i} = 0. \tag{1.15}$$

Assume further that $\lambda_n := \sum_{i=1}^{m_n} p_{n,i} \to \gamma$ as $n \to \infty$, where $\gamma > 0$, and that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \mathbb{P}(X_{n,i} \geq 2) = 0. \tag{1.16}$$

Let $X_n := \sum_{i=1}^{m_n} X_{n,i}$. Then for $k \in \mathbb{N}_0$ we have

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \mathrm{Po}(\gamma; k). \tag{1.17}$$

Proof Let $X'_{n,i} := \mathbf{1}\{X_{n,i} \geq 1\} = \min\{X_{n,i}, 1\}$ and $X'_n := \sum_{i=1}^{m_n} X'_{n,i}$. Since $X'_{n,i} \neq X_{n,i}$ if and only if $X_{n,i} \geq 2$, we have

$$\mathbb{P}(X'_n \neq X_n) \leq \sum_{i=1}^{m_n} \mathbb{P}(X_{n,i} \geq 2).$$

By assumption (1.16) we can assume without restriction of generality that

$X'_{n,i} = X_{n,i}$ for all $n \in \mathbb{N}$ and $i \in \{1, \dots, m_n\}$. Moreover it is no loss of generality to assume for each (n, i) that $p_{n,i} < 1$. We then have

$$\mathbb{P}(X_n = k) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} \frac{\prod_{j=1}^{m_n}(1 - p_{n,j})}{(1 - p_{n,i_1}) \cdots (1 - p_{n,i_k})}. \tag{1.18}$$

Let $\mu_n := \max_{1 \le i \le m_n} p_{n,i}$. Since $\sum_{j=1}^{m_n} p_{n,j}^2 \le \lambda_n \mu_n \to 0$ as $n \to \infty$, we have

$$\log\left(\prod_{j=1}^{m_n}(1 - p_{n,j})\right) = \sum_{j=1}^{m_n}(-p_{n,j} + O(p_{n,j}^2)) \to -\gamma \text{ as } n \to \infty, \tag{1.19}$$

where the function $O(\cdot)$ satisfies $\limsup_{r \to 0} |r|^{-1}|O(r)| < \infty$. Also,

$$\inf_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} (1 - p_{n,i_1}) \cdots (1 - p_{n,i_k}) \ge (1 - \mu_n)^k \to 1 \text{ as } n \to \infty. \tag{1.20}$$

Finally, with $\sum^{\neq}_{i_1,\dots,i_k \in \{1,2,\dots,m_n\}}$ denoting summation over all ordered k-tuples of distinct elements of $\{1, 2, \dots, m_n\}$, we have

$$k! \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} = \sum^{\neq}_{i_1,\dots,i_k \in \{1,2,\dots,m_n\}} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k},$$

and

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{m_n} p_{n,i}\right)^k - \sum^{\neq}_{i_1,\dots,i_k \in \{1,2,\dots,m_n\}} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} \\ &\le \binom{k}{2} \sum_{i=1}^{m_n} p_{n,i}^2 \left(\sum_{j=1}^{m_n} p_{n,j}\right)^{k-2}, \end{aligned}$$

which tends to zero as $n \to \infty$. Therefore

$$k! \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} \to \gamma^k \text{ as } n \to \infty. \tag{1.21}$$

The result follows from (1.18) by using (1.19), (1.20) and (1.21). □

1.4 The Negative Binomial Distribution

A random element Z of $\mathbb{N}_0$ is said to have a *negative binomial distribution* with parameters $r > 0$ and $p \in (0, 1]$ if

$$\mathbb{P}(Z = n) = \frac{\Gamma(n + r)}{\Gamma(n + 1)\Gamma(r)}(1 - p)^n p^r, \quad n \in \mathbb{N}_0, \tag{1.22}$$

where the *Gamma function* $\Gamma\colon (0,\infty) \to (0,\infty)$ is defined by

$$\Gamma(a) := \int_0^\infty t^{a-1} e^{-t}\, dt, \quad a > 0. \tag{1.23}$$

(In particular $\Gamma(a) = (a-1)!$ for $a \in \mathbb{N}$.) This can be seen to be a probability distribution by Taylor expansion of $(1-x)^{-r}$ evaluated at $x = 1 - p$. The probability generating function of Z is given by

$$\mathbb{E}[s^Z] = p^r (1 - s + sp)^{-r}, \quad s \in [0,1]. \tag{1.24}$$

For $r \in \mathbb{N}$, such a Z may be interpreted as the number of failures before the rth success in a sequence of independent Bernoulli trials. In the special case $r = 1$ we get the *geometric distribution*

$$\mathbb{P}(Z = n) = (1-p)^n p, \quad n \in \mathbb{N}_0. \tag{1.25}$$

Another interesting special case is $r = 1/2$. In this case

$$\mathbb{P}(Z = n) = \frac{(2n-1)!!}{2^n n!} (1-p)^n p^{1/2}, \quad n \in \mathbb{N}_0, \tag{1.26}$$

where we recall the definition (B.6) for $(2n-1)!!$. This follows from the fact that $\Gamma(n + 1/2) = (2n-1)!!\, 2^{-n} \sqrt{\pi}$, $n \in \mathbb{N}_0$.

The negative binomial distribution arises as a mixture of Poisson distributions. To explain this, we need to introduce the *Gamma distribution* with *shape parameter* $a > 0$ and *scale parameter* $b > 0$. This is a probability measure on $\mathbb{R}_+$ with Lebesgue density

$$x \mapsto b^a \Gamma(a)^{-1} x^{a-1} e^{-bx} \tag{1.27}$$

on $\mathbb{R}_+$. If a random variable Y has this distribution, then one says that Y is Gamma distributed with shape parameter a and scale parameter b. In this case Y has Laplace transform

$$\mathbb{E}[e^{-tY}] = \left(\frac{b}{b+t}\right)^a, \quad t \geq 0. \tag{1.28}$$

In the case $a = 1$ we obtain the *exponential distribution* with parameter b. Exercise 1.11 asks the reader to prove the following result.

Proposition 1.5 *Suppose that the random variable $Y \geq 0$ is Gamma distributed with shape parameter $a > 0$ and scale parameter $b > 0$. Let Z be an $\mathbb{N}_0$-valued random variable such that the conditional distribution of Z given Y is* $\mathrm{Po}(Y)$. *Then Z has a negative binomial distribution with parameters a and $b/(b+1)$.*

1.5 Exercises

Exercise 1.1 Prove equation (1.10).

Exercise 1.2 Let X be a random variable taking values in $\mathbb{N}_0$. Assume that there is a $\gamma \geq 0$ such that $\mathbb{E}[(X)_k] = \gamma^k$ for all $k \in \mathbb{N}_0$. Show that X has a Poisson distribution. (Hint: Derive the Taylor series for $g(s) := \mathbb{E}[s^X]$ at $s_0 = 1$.)

Exercise 1.3 Confirm Proposition 1.3 by showing that

$$\mathbb{E}[s^X t^{Z-X}] = e^{p\gamma(s-1)} e^{(1-p)\gamma(t-1)}, \quad s, t \in [0, 1],$$

using a direct computation and Proposition B.4.

Exercise 1.4 (Generalisation of Proposition 1.2) Let $m \in \mathbb{N}$ and suppose that $X_1, \ldots, X_m$ are independent random variables with Poisson distributions $\mathrm{Po}(\gamma_1), \ldots, \mathrm{Po}(\gamma_m)$, respectively. Show that $X := X_1 + \cdots + X_m$ is Poisson distributed with parameter $\gamma := \gamma_1 + \cdots + \gamma_m$. Assuming $\gamma > 0$, show moreover for any $k \in \mathbb{N}$ that

$$\mathbb{P}(X_1 = k_1, \ldots, X_m = k_m \mid X = k) = \frac{k!}{k_1! \cdots k_m!} \left(\frac{\gamma_1}{\gamma}\right)^{k_1} \cdots \left(\frac{\gamma_m}{\gamma}\right)^{k_m} \tag{1.29}$$

for $k_1 + \cdots + k_m = k$. This is a *multinomial distribution* with parameters k and $\gamma_1/\gamma, \ldots, \gamma_m/\gamma$.

Exercise 1.5 (Generalisation of Proposition 1.3) Let $m \in \mathbb{N}$ and suppose that Z_n, $n \in \mathbb{N}$, is a sequence of independent random vectors in $\mathbb{R}^m$ with common distribution $\mathbb{P}(Z_1 = e_i) = p_i$, $i \in \{1, \ldots, m\}$, where e_i is the i-th unit vector in $\mathbb{R}^m$ and $p_1 + \cdots + p_m = 1$. Let Z have a Poisson distribution with parameter γ, independent of $(Z_1, Z_2, \ldots)$. Show that the components of the random vector $X := \sum_{j=1}^{Z} Z_j$ are independent and Poisson distributed with parameters $p_1\gamma, \ldots, p_m\gamma$.

Exercise 1.6 (Bivariate extension of Proposition 1.4) Let $\gamma > 0, \delta \geq 0$. Suppose for $n \in \mathbb{N}$ that $m_n \in \mathbb{N}$ and for $1 \leq i \leq m_n$ that $p_{n,i}, q_{n,i} \in [0, 1)$ with $\sum_{i=1}^{m_n} p_{n,i} \to \gamma$ and $\sum_{i=1}^{m_n} q_{n,i} \to \delta$, and $\max_{1 \leq i \leq m_n} \max\{p_{n,i}, q_{n,i}\} \to 0$ as $n \to \infty$. Suppose for $n \in \mathbb{N}$ that $(X_n, Y_n) = \sum_{i=1}^{m_n} (X_{n,i}, Y_{n,i})$, where each $(X_{n,i}, Y_{n,i})$ is a random 2-vector whose components are Bernoulli distributed with parameters $p_{n,i}, q_{n,i}$, respectively, and satisfy $X_{n,i} Y_{n,i} = 0$ almost surely. Assume the random vectors $(X_{n,i}, Y_{n,i})$, $1 \leq i \leq m_n$, are independent. Prove that X_n, Y_n are asymptotically (as $n \to \infty$) distributed as a pair of independent Poisson variables with parameters γ, δ, i.e. for $k, \ell \in \mathbb{N}_0$,

$$\lim_{n\to\infty} \mathbb{P}(X_n = k, Y_n = \ell) = e^{-(\gamma+\delta)} \frac{\gamma^k \delta^\ell}{k!\ell!}.$$

Exercise 1.7 (Probability of a Poisson variable being even) Suppose X is Poisson distributed with parameter $\gamma > 0$. Using the fact that the probability generating function (1.8) extends to $s = -1$, verify the identity $\mathbb{P}(X/2 \in \mathbb{Z}) = (1 + e^{-2\gamma})/2$. For $k \in \mathbb{N}$ with $k \geq 3$, using the fact that the probability generating function (1.8) extends to a k-th complex root of unity, find a closed-form formula for $\mathbb{P}(X/k \in \mathbb{Z})$.

Exercise 1.8 Let $\gamma > 0$, and suppose X is Poisson distributed with parameter γ. Suppose $f : \mathbb{N} \to \mathbb{R}_+$ is such that $\mathbb{E}[f(X)^{1+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Show that $\mathbb{E}[f(X + k)] < \infty$ for any $k \in \mathbb{N}$.

Exercise 1.9 Let $0 < \gamma < \gamma'$. Give an example of a random vector (X, Y) with X Poisson distributed with parameter γ and Y Poisson distributed with parameter γ', such that $Y - X$ *is not* Poisson distributed. (Hint: First consider a pair X', Y' such that $Y' - X'$ *is* Poisson distributed, and then modify finitely many of the values of their joint probability mass function.)

Exercise 1.10 Suppose $n \in \mathbb{N}$ and set $[n] := \{1, \dots, n\}$. Suppose that Z is a uniform random permutation of $[n]$, that is a random element of the space Σ_n of all bijective mappings from $[n]$ to $[n]$ such that $\mathbb{P}(Z = \pi) = 1/n!$ for each $\pi \in \Sigma_n$. For $a \in \mathbb{R}$ let $\lceil a \rceil := \min\{k \in \mathbb{Z} : k \geq a\}$. Let $\gamma \in [0, 1]$ and let $X_n := \mathrm{card}\{i \in [\lceil \gamma n \rceil] : Z(i) = i\}$ be the number of fixed points of Z among the first $\lceil \gamma n \rceil$ integers. Show that the distribution of X_n converges to $\mathrm{Po}(\gamma)$, that is

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \frac{\gamma^k}{k!} e^{-\gamma}, \quad k \in \mathbb{N}_0.$$

(Hint: Establish an explicit formula for $\mathbb{P}(X_n = k)$, starting with the case $k = 0$.)

Exercise 1.11 Prove Proposition 1.5.

Exercise 1.12 Let $\gamma > 0$ and $\delta > 0$. Find a random vector (X, Y) such that X, Y and $X + Y$ are Poisson distributed with parameter γ, δ and $\gamma + \delta$, respectively, but X and Y are not independent.

2

Point Processes

A point process is a random collection of at most countably many points, possibly with multiplicities. This chapter defines this concept for an arbitrary measurable space and provides several criteria for equality in distribution.

2.1 Fundamentals

The idea of a point process is that of a random, at most countable, collection Z of points in some space $\mathbb{X}$. A good example to think of is the d-dimensional Euclidean space $\mathbb{R}^d$. Ignoring measurability issues for the moment, we might think of Z as a mapping $\omega \mapsto Z(\omega)$ from Ω into the system of countable subsets of $\mathbb{X}$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is an underlying probability space. Then Z can be identified with the family of mappings

$$\omega \mapsto \eta(\omega, B) := \mathrm{card}(Z(\omega) \cap B), \qquad B \subset \mathbb{X},$$

counting the number of points that Z has in B. (We write $\mathrm{card}\, A$ for the number of elements of a set A.) Clearly, for any fixed $\omega \in \Omega$ the mapping $\eta(\omega, \cdot)$ is a measure, namely the *counting measure* supported by $Z(\omega)$. It turns out to be a mathematically fruitful idea to define point processes as random counting measures.

To give the general definition of a point process let $(\mathbb{X}, \mathcal{X})$ be a measurable space. Let $\mathbf{N}_{<\infty}(\mathbb{X}) \equiv \mathbf{N}_{<\infty}$ denote the space of all measures μ on $\mathbb{X}$ such that $\mu(B) \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for all $B \in \mathcal{X}$, and let $\mathbf{N}(\mathbb{X}) \equiv \mathbf{N}$ be the space of all measures that can be written as a countable sum of measures from $\mathbf{N}_{<\infty}$. A trivial example of an element of $\mathbf{N}$ is the *zero measure* 0 that is identically zero on $\mathcal{X}$. A less trivial example is the *Dirac measure* δ_x at a point $x \in \mathbb{X}$ given by $\delta_x(B) := \mathbf{1}_B(x)$. More generally, any (finite or infinite) sequence $(x_n)_{n=1}^k$ of elements of $\mathbb{X}$, where $k \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ is the number of terms in the sequence, can be used to define a measure

$$\mu = \sum_{n=1}^{k} \delta_{x_n}. \tag{2.1}$$

Then $\mu \in \mathbf{N}$ and

$$\mu(B) = \sum_{n=1}^{k} \mathbf{1}_B(x_n), \quad B \in \mathcal{X}.$$

More generally we have, for any measurable $f : \mathbb{X} \to [0, \infty]$, that

$$\int f \, d\mu = \sum_{n=1}^{k} f(x_n). \tag{2.2}$$

We can allow for $k = 0$ in (2.1). In this case μ is the zero measure. The points $x_1, x_2, \ldots$ are not assumed to be pairwise distinct. If $x_i = x_j$ for some $i, j \leq k$ with $i \neq j$, then μ is said to have *multiplicities*. In fact, the multiplicity of x_i is the number $\mathrm{card}\{j \leq k : x_j = x_i\}$. Any μ of the form (2.1) is interpreted as a counting measure with possible multiplicities.

In general one cannot guarantee that any $\mu \in \mathbf{N}$ can be written in the form (2.1); see Exercise 2.5. Fortunately, only weak assumptions on $(\mathbb{X}, \mathcal{X})$ and μ are required to achieve this; see e.g. Corollary 6.5. Moreover, large parts of the theory can be developed without imposing further assumptions on $(\mathbb{X}, \mathcal{X})$, other than to be a measurable space.

A measure ν on $\mathbb{X}$ is said to be *s-finite* if ν is a countable sum of finite measures. By definition, each element of $\mathbf{N}$ is s-finite. We recall that a measure ν on $\mathbb{X}$ is said to be *σ-finite* if there is a sequence $B_m \in \mathcal{X}$, $m \in \mathbb{N}$, such that $\cup_m B_m = \mathbb{X}$ and $\nu(B_m) < \infty$ for all $m \in \mathbb{N}$. Clearly every σ-finite measure is s-finite. Any $\overline{\mathbb{N}}_0$-valued σ-finite measure is in $\mathbf{N}$. In contrast to σ-finite measures, any countable sum of s-finite measures is again s-finite. If the points x_n in (2.1) are all the same, then this measure μ is not σ-finite. The counting measure on $\mathbb{R}$ (supported by $\mathbb{R}$) is an example of a measure with values in $\overline{\mathbb{N}}_0 := \overline{\mathbb{N}} \cup \{0\}$, that is not s-finite. Exercise 6.10 gives an example of an s-finite $\overline{\mathbb{N}}_0$-valued measure that is not in $\mathbf{N}$.

Let $\mathcal{N}(\mathbb{X}) \equiv \mathcal{N}$ denote the σ-field generated by the collection of all subsets of $\mathbf{N}$ of the form

$$\{\mu \in \mathbf{N} : \mu(B) = k\}, \quad B \in \mathcal{X},\ k \in \mathbb{N}_0.$$

This means that $\mathcal{N}$ is the smallest σ-field on $\mathbf{N}$ such that $\mu \mapsto \mu(B)$ is measurable for all $B \in \mathcal{X}$.