A Concise Text on Advanced Linear Algebra

This engaging textbook for advanced undergraduate students and beginning graduates covers the core subjects in linear algebra. The author motivates the concepts by drawing clear links to applications and other important areas.

The book places particular emphasis on integrating ideas from analysis wherever appropriate and features many novelties in its presentation. For example, the notion of determinant is shown to appear from calculating the index of a vector field which leads to a self-contained proof of the Fundamental Theorem of Algebra; the Cayley–Hamilton theorem is established by recognizing the fact that the set of complex matrices of distinct eigenvalues is dense; the existence of a real eigenvalue of a self-adjoint map is deduced by the method of calculus; the construction of the Jordan decomposition is seen to boil down to understanding nilpotent maps of degree two; and a lucid and elementary introduction to quantum mechanics based on linear algebra is given.

The material is supplemented by a rich collection of over 350 mostly proof-oriented exercises, suitable for readers from a wide variety of backgrounds. Selected solutions are provided at the back of the book, making it ideal for self-study as well as for use as a course text.

A Concise Text on Advanced Linear Algebra

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CAMBRIDGE
UNIVERSITY PRESS

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University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107087514

First published 2015

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Yang, Yisong.
A concise text on advanced linear algebra / Yisong Yang, Polytechnic School of Engineering, New York University.
pages cm
Includes bibliographical references and index.
ISBN 978-1-107-08751-4 (Hardback) – ISBN 978-1-107-45681-5 (Paperback)
1. Algebras, Linear–Textbooks. 2. Algebras, Linear–Study and teaching (Higher) .
3. Algebras, Linear–Study and teaching (Graduate). I. Title.
II. Title: Advanced linear algebra.
QA184.2.Y36 2015
512′.5–dc23 2014028951

ISBN 978-1-107-08751-4 Hardback
ISBN 978-1-107-45681-5 Paperback

For Sheng,
Peter, Anna, and Julia

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Preface

This book is concisely written to provide comprehensive core materials for a year-long course in Linear Algebra for senior undergraduate and beginning graduate students in mathematics, science, and engineering. Students who gain profound understanding and grasp of the concepts and methods of this course will acquire an essential knowledge foundation to excel in their future academic endeavors.

Throughout the book, methods and ideas of analysis are greatly emphasized and used, along with those of algebra, wherever appropriate, and a delicate balance is cast between abstract formulation and practical origins of various subject matters.

The book is divided into nine chapters. The first seven chapters embody a traditional course curriculum. An outline of the contents of these chapters is sketched as follows.

In Chapter 1 we cover basic facts and properties of vector spaces. These include definitions of vector spaces and subspaces, concepts of linear dependence, bases, coordinates, dimensionality, dual spaces and dual bases, quotient spaces, normed spaces, and the equivalence of the norms of a finite-dimensional normed space.

In Chapter 2 we cover linear mappings between vector spaces. We start from the definition of linear mappings and discuss how linear mappings may be concretely represented by matrices with respect to given bases. We then introduce the notion of adjoint mappings and quotient mappings. Linear mappings from a vector space into itself comprise a special but important family of mappings and are given a separate treatment later in this chapter. Topics studied there include invariance and reducibility, eigenvalues and eigenvectors, projections, nilpotent mappings, and polynomials of linear mappings. We end the chapter with a discussion of the concept of the norms of linear mappings and use it to show that being invertible is a generic property of a linear mapping and then to show how the exponential of a linear mapping may be constructed and understood.

In Chapter 3 we cover determinants. As a non-traditional but highly motivating example, we show that the calculation of the topological degree of a differentiable map from a closed curve into the unit circle in $\mathbb{R}^2$ involves computing a two-by-two determinant, and the knowledge gained allows us to prove the Fundamental Theorem of Algebra. We then formulate the definition of a general determinant inductively, without resorting to the notion of permutations, and establish all its properties. We end the chapter by establishing the Cayley–Hamilton theorem. Two independent proofs of this important theorem are given. The first proof is analytic and consists of two steps. In the first step, we show that the theorem is valid for a matrix of distinct eigenvalues. In the second step, we show that any matrix may be regarded as a limiting point of a sequence of matrices of distinct eigenvalues. Hence the theorem follows again by taking the limit. The second proof, on the other hand, is purely algebraic.

In Chapter 4 we discuss vector spaces with scalar products. We start from the most general notion of scalar products without requiring either non-degeneracy or positive definiteness. We then carry out detailed studies on non-degenerate and positive definite scalar products, respectively, and elaborate on adjoint mappings in terms of scalar products. We end the chapter with a discussion of isometric mappings in both real and complex space settings and noting their subtle differences.

In Chapter 5 we focus on real vector spaces with positive definite scalar products and quadratic forms. We first establish the main spectral theorem for self-adjoint mappings. We will not take the traditional path of first using the Fundamental Theorem of Algebra to assert that there is an eigenvalue and then applying the self-adjointness to show that the eigenvalue must be real. Instead we shall formulate an optimization problem and use calculus to prove directly that a self-adjoint mapping must have a real eigenvalue. We then present a series of characteristic conditions for a symmetric bilinear form, a symmetric matrix, or a self-adjoint mapping, to be positive definite. We end the chapter by a discussion of the commutativity of self-adjoint mappings and the usefulness of self-adjoint mappings for the investigation of linear mappings between different spaces.

In Chapter 6 we study complex vector spaces with Hermitian scalar products and related notions. Much of the theory here is parallel to that of the real space situation with the exception that normal mappings can only be fully understood and appreciated within a complex space formalism.

In Chapter 7 we establish the Jordan decomposition theorem. We start with a discussion of some basic facts regarding polynomials. We next show how

Cambridge University Press
978-1-107-08751-4 - A Concise Text on Advanced Linear Algebra
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to reduce a linear mapping over its generalized eigenspaces via the Cayley–Hamilton theorem and the prime factorization of the characteristic polynomial of the mapping. We then prove the Jordan decomposition theorem. The key and often the most difficult step in this construction is a full understanding of how a nilpotent mapping is reduced canonically. We approach this problem inductively with the degree of a nilpotent mapping and show that it is crucial to tackle a mapping of degree 2. Such a treatment eases the subtlety of the subject considerably.

In Chapter 8 we present four selected topics that may be used as materials for some optional extra-curricular study when time and interest permit. In the first section we present the Schur decomposition theorem, which may be viewed as a complement to the Jordan decomposition theorem. In the second section we give a classification of skewsymmetric bilinear forms. In the third section we state and prove the Perron–Frobenius theorem regarding the principal eigenvalues of positive matrices. In the fourth section we establish some basic properties of the Markov matrices.

In Chapter 9 we present yet another selected topic for the purpose of optional extra-curricular study: a short excursion into quantum mechanics using gadgets purely from linear algebra. Specifically we will use $\mathbb{C}^n$ as the state space and Hermitian matrices as quantum mechanical observables to formulate the over-simplified quantum mechanical postulates including Bohr's statistical interpretation of quantum mechanics and the Schrödinger equation governing the time evolution of a state. We next establish Heisenberg's uncertainty principle. Then we prove the equivalence of the Schrödinger description via the Schrödinger equation and the Heisenberg description via the Heisenberg equation of quantum mechanics.

Also provided in the book is a rich collection of mostly proof-oriented exercises to supplement and consolidate the main course materials. The diversity and elasticity of these exercises aim to satisfy the needs and interests of students from a wide variety of backgrounds.

At the end of the book, solutions to some selected exercises are presented. These exercises and solutions provide additional illustrative examples, extend main course materials, and render convenience for the reader to master the subjects and methods covered in a broader range.

Finally some bibliographic notes conclude the book.

This text may be curtailed to meet the time constraint of a semester-long course. Here is a suggested list of selected sections for such a plan: Sections 1.1–1.5, 2.1–2.3, 2.5, 3.1.2, 3.2, and 3.3 (present the concept of adjugate matrices only), Section 3.4 (give the second proof of the Cayley–Hamilton theorem only, based on an adjugate matrix expansion), Sections 4.3, 4.4, 5.1, 5.2

(omit the analytic proof that a self-adjoint mapping must have an eigenvalue but resort to Exercise 5.2.1 instead), Sections 5.3, 6.1, 6.2, 6.3.1, and 7.1–7.4. Depending on the pace of lectures and time available, the instructor may decide in the later stage of the course to what extent the topics in Sections 7.1–7.4 (the Jordan decomposition) can be presented productively.

The author would like to take this opportunity to thank Patrick Lin, Thomas Otway, and Robert Sibner for constructive comments and suggestions, and Roger Astley of Cambridge University Press for valuable editorial advice, which helped improve the presentation of the book.

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Notation and convention

We use $\mathbb{N}$ to denote the set of all natural numbers,

$$\mathbb{N} = \{0, 1, 2, \dots\},$$

and $\mathbb{Z}$ the set of all integers,

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

We use i to denote the imaginary unit $\sqrt{-1}$. For a complex number $c = a + \mathrm{i}b$ where a, b are real numbers we use

$$\overline{c} = a - \mathrm{i}b$$

to denote the complex conjugate of c. We use $\Re\{c\}$ and $\Im\{c\}$ to denote the real and imaginary parts of the complex number $c = a + \mathrm{i}b$. That is,

$$\Re\{c\} = a, \quad \Im\{c\} = b.$$

We use i, j, k, l, m, n to denote integer-valued indices or space dimension numbers, a, b, c scalars, u, v, w, x, y, z vectors, A, B, C, D matrices, P, R, S, T mappings, and U, V, W, X, Y, Z vector spaces, unless otherwise stated.

We use t to denote the variable in a polynomial or a function or the transpose operation on a vector or a matrix.

When X or Y is given, we use $X \equiv Y$ to denote that Y, or X, is defined to be X, or Y, respectively.

Occasionally, we use the symbol $\forall$ to express 'for all'.

Let X be a set and Y, Z subsets of X. We use $Y \setminus Z$ to denote the subset of elements in Y which are not in Z.