Pseudo-reductive Groups
Second Edition

Pseudo-reductive groups arise naturally in the study of general smooth linear algebraic groups over non-perfect fields and have many important applications. This monograph provides a comprehensive treatment of the theory of pseudo-reductive groups and explains their structure in a usable form.

In this second edition there is new material on relative root systems and Tits systems for general smooth affine groups, including the extension to quasi-reductive groups of famous simplicity results of Tits in the semisimple case. Chapter 9 has been completely rewritten to describe and classify pseudo-split absolutely pseudo-simple groups with a non-reduced root system over arbitrary fields of characteristic 2 via the useful new notion of “minimal type” for pseudo-reductive groups.

Researchers and graduate students working in related areas such as algebraic geometry, algebraic group theory, or number theory will value this book, as it develops tools likely to be used in tackling other problems.

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Dedicated to Jacques Tits

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Preface to the second edition

In addition to correcting minor errors/misprints and simplifying some proofs as well as improving some results, the major changes in this edition are: a complete rewriting of Chapter 9 (to obtain definitive results over all fields of characteristic 2), a simplified and improved exposition of Tits' results on unipotent groups given in Appendix B, and additional material C.2.11–C.2.34 in Appendix C to provide a version of the Borel–Tits relative structure theory (with relative root systems, etc.) in the pseudo-reductive case and beyond. Apart from Chapter 9, parts of §11.4, and the final displayed expression in §1.6, all numerical labels for results, examples, equations, remarks, etc. remain identical to the labels in the first edition (but the formulation of some results has been strengthened and new material has been added at the end of some sections).

We recall that pseudo-split pseudo-simple groups with non-reduced root systems can only exist over imperfect fields k of characteristic 2. In Chapter 9 of the original version of this monograph, we constructed such groups and explored their properties only when $[k : k^2] = 2$. In the revised Chapter 9, we introduce some new ideas (especially the property that we call "minimal type") and use them to eliminate the degree restriction, yielding a definitive treatment of such groups; when $[k : k^2] > 2$ there arise several new phenomena with no analogue when $[k : k^2] = 2$. For the convenience of the reader we have retained the hypothesis $[k : k^2] = 2$ for the classification results in Chapter 10, but the results in this new edition (especially in Chapter 9) are used in [CP] to study the automorphism groups of pseudo-semisimple groups and give a complete classification of pseudo-reductive groups (in the spirit of Chapter 10) without any restriction on $[k : k^2]$ when $\mathrm{char}(k) = 2$.

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Introduction

Why go beyond reductive groups?

The theory of connected reductive groups over a general field, and its applications over arithmetically interesting fields, constitutes one of the most beautiful topics within pure mathematics. However, it does sometimes happen that one is confronted with linear algebraic groups that are not reductive, and whose structure may be rather mysterious.

Example: *forms of a variety*. Let X be a projective variety over a field k. Grothendieck constructed a scheme $\mathrm{Aut}_{X/k}$ classifying its automorphisms, but this is hard to understand in general. For example, finite generation of its component group is unknown, even if $k = \mathbf{C}$. Likewise, little is known about the identity component $\mathrm{Aut}^0_{X/k}$ apart from that it is a k-group scheme of finite type. Since $\mathrm{H}^1(k, \mathrm{Aut}_{X/k})$ classifies k-forms of X, there is arithmetic interest in $\mathrm{Aut}_{X/k}$ even though our knowledge of its structure is limited.

If $\mathrm{char}(k) = 0$ then by a structure theorem of Chevalley for smooth connected groups over perfect fields, $\mathrm{Aut}^0_{X/k}$ is an extension of an abelian variety by a smooth connected affine k-group. If k is imperfect then Chevalley's Theorem does not apply (even if $\mathrm{Aut}^0_{X/k}$ is smooth). Nonetheless, some general problems for connected k-group schemes G of finite type (e.g., $G = \mathrm{Aut}^0_{X/k}$) reduce to the cases of smooth connected affine k-groups and abelian varieties over k. We do not know any restrictions on the smooth connected affine groups arising in this way when $G = \mathrm{Aut}^0_{X/k}$.

Example: *local-to-global principle*. Let X be a quasi-projective (or arbitrary) scheme over a global field k. Suppose that X is equipped with a right action by a linear algebraic k-group H. Choose a point $x \in X(k)$. Does $H(k)$ act with only finitely many orbits on the set of $x' \in X(k)$ that are $H(k_v)$-conjugate to x for all v away from a fixed finite set S of places of k?

This question reduces to the finiteness of the Tate–Shafarevich sets $Ш^1_S(k, G)$ (of isomorphism classes of right G-torsors over k admitting a k_v-point for all $v \notin S$) as G varies through the isotropy group schemes H_x of all $x \in X(k)$. Such isotropy groups are generally disconnected and non-reductive (and non-smooth if $\mathrm{char}(k) > 0$), even if H is connected and semisimple. Over number fields k the finiteness of $Ш^1_S(k, G)$ for any linear algebraic k-group G was proved by Borel and Serre [BoSe, Thm. 7.1]. The analogous result over global function fields reduces to the case of smooth G (see Example C.4.3) and the results in this monograph provide what is needed to settle this case.

Imperfect base fields

Over fields of characteristic 0, or more generally over perfect fields (such as finite fields), it is usually an elementary matter to reduce problems for general linear algebraic groups to the connected reductive case. The situation is entirely different over imperfect fields, such as local and global function fields. The purpose of the theory of pseudo-reductive groups, initiated by Borel and Tits, is to overcome this problem over imperfect fields.

To explain the reason for the difficulties over imperfect fields, consider a smooth connected affine group G over an arbitrary field k. Recall that the unipotent radical $\mathscr{R}_u(G_{\overline{k}})$ over an algebraic closure $\overline{k}$ of k is the unique maximal smooth connected unipotent normal $\overline{k}$-subgroup of $G_{\overline{k}}$, and that G is *reductive* if $\mathscr{R}_u(G_{\overline{k}}) = 1$. If k is perfect then $\overline{k}/k$ is Galois, so by Galois descent $\mathscr{R}_u(G_{\overline{k}})$ (uniquely) descends to a smooth connected unipotent normal k-subgroup $\mathscr{R}_u(G) \subseteq G$. In particular, for perfect k there is an exact sequence of k-groups

$$1 \to \mathscr{R}_u(G) \to G \to G/\mathscr{R}_u(G) \to 1 \qquad (*)$$

with $G/\mathscr{R}_u(G)$ a connected reductive k-group. For imperfect k the unipotent radical $\mathscr{R}_u(G_{\overline{k}})$ can fail to descend to a k-subgroup of G (as we will illustrate with many examples), in which case G is not an extension of a connected reductive k-group by a smooth connected unipotent k-group.

The *k-unipotent radical* $\mathscr{R}_{u,k}(G)$ is the maximal smooth connected unipotent normal k-subgroup of G. For an extension field k'/k, the natural inclusion

$$k' \otimes_k \mathscr{R}_{u,k}(G) \hookrightarrow \mathscr{R}_{u,k'}(G_{k'})$$

is an equality if k'/k is separable but can fail to be an equality otherwise (e.g., for $k' = \overline{k}$ when k is imperfect). Over a global field k there are arithmetic problems (such as the ones mentioned above) that can be reduced to questions about smooth connected affine k-groups H for which the inclusion $\overline{k} \otimes_k \mathscr{R}_{u,k}(H) \hookrightarrow \mathscr{R}_u(H_{\overline{k}})$ may not be an equality when $\text{char}(k) > 0$.

A *pseudo-reductive group* over a field k is a smooth connected affine k-group with trivial k-unipotent radical. For any smooth connected affine k-group G, the exact sequence

$$1 \to \mathscr{R}_{u,k}(G) \to G \to G/\mathscr{R}_{u,k}(G) \to 1$$

uniquely expresses G as an extension of a pseudo-reductive k-group by a smooth connected unipotent k-group. (For perfect k this is ($*$).) To use this exact sequence, we need a structure theory for pseudo-reductive groups that is as informative as the one for connected reductive groups.

Structure theory

A useful structure theory for pseudo-reductive groups over imperfect fields cannot be developed by minor variations of the methods used in the reductive case, essentially because pseudo-reductivity is not a geometric property. More specifically, pseudo-reductivity can be lost under an inseparable extension of the base field, such as extension from an imperfect field to an algebraically closed field. Also, pseudo-reductivity is not inherited by quotients in general, as will be seen in many examples. It seems unlikely that there can be an interesting theory of pseudo-reductive groups over a base ring other than a field, in contrast with the theory of connected reductive groups.

Pseudo-reductive groups were the topic of courses by J. Tits at the Collège de France in 1991–92 and 1992–93; see [Ti2] and [Ti3]. In these courses, he studied pseudo-reductive groups over any field k in terms of pseudo-parabolic k-subgroups and a root datum, and he constructed "non-standard" examples of pseudo-reductive groups over separably closed fields of characteristics 2 and 3.

Most of Tits' results in his second course were announced by Borel and Tits in [BoTi2] without proofs. Tits' résumé [Ti3] sketched some of the proofs, and a more detailed exposition was given by Springer in [Spr, Ch. 13–15]. Tits also attempted to classify pseudo-reductive groups over separably closed fields of nonzero characteristic, but he ran into many complications and so did not write up all of his results. For example, the structure of root groups was shrouded in mystery; their dimension can be rather large. Inspired by [BoTi2], Springer obtained (unpublished) classification results in cases with geometric semisimple rank 1, and he made partial progress in characteristic 2.

We discovered a new approach to the structure of pseudo-reductive groups. If k is a field with $\mathrm{char}(k) \neq 2$ then our main result over k is a canonical description of all pseudo-reductive k-groups in terms of two ingredients: connected semisimple groups *over finite extensions* of k, and *commutative* pseudo-reductive k-groups. (These commutative groups turn out to be the Cartan k-subgroups, which are the maximal k-tori in the connected reductive case.) We also get a similar result when $\mathrm{char}(k) = 2$ if k is "almost perfect" in the sense that $[k : k^2] \leqslant 2$. This includes the most interesting cases for number theory: local and global function fields over finite fields. The case of imperfect fields of characteristic 2 involves new phenomena. Most of the hardest aspects of characteristic 2 show up in the study of G whose maximal geometric semisimple quotient $G^{\mathrm{ss}}_{\overline{k}}$ is SL_2 or PGL_2.

Main results and applications

There is a lot of "practical" interest in connected reductive groups (and especially connected semisimple groups), due to applications in the study of interesting problems not originating from the theory of algebraic groups. The reason for interest in pseudo-reductive groups is largely theoretical: they arise naturally in proofs of general theorems. Whenever one is faced with proving a result for arbitrary smooth connected affine groups over an imperfect field (e.g., reducing a problem for a non-smooth affine group to an analogous problem for a related abstract smooth affine group by the method of §C.4), pseudo-reductive groups show up almost immediately whereas reductive groups do not. More importantly, experience shows that the structure theory of pseudo-reductive groups often makes it possible to reduce general problems to the semisimple and solvable cases. This is why we believe that pseudo-reductive groups are basic objects of interest in the theory of linear algebraic groups over imperfect fields.

Our main discovery is that, apart from some exceptional situations in characteristics 2 and 3, all pseudo-reductive groups can be canonically constructed in terms of three basic methods: direct products, Weil restriction of connected semisimple groups over *finite extensions of* k, and modifying a Cartan subgroup (via a central pushout construction).

To explain how this works, first consider a finite extension of fields k'/k and a smooth connected affine k'-group G'. The Weil restriction $G = \mathrm{R}_{k'/k}(G')$ is a smooth connected affine k-group of dimension $[k' : k] \dim G'$ characterized by the property $G(A) = G'(k' \otimes_k A)$ for k-algebras A. When k'/k is separable, the functor $\mathrm{R}_{k'/k}$ has good properties and is very useful. For example, it preserves reductivity, and any connected semisimple k-group G that is k-simple and simply connected has the form $\mathrm{R}_{k'/k}(G')$ for a canonically associated pair $(G', k'/k)$ where k'/k is a separable extension and G' is *absolutely simple* and simply connected over k'.

The functor $\mathrm{R}_{k'/k}$ exhibits some unusual properties when k'/k is not separable: it does not carry tori to tori, nor does it preserve reductivity or perfectness of groups or properness of morphisms in general, and it often fails to preserve surjectivity or finiteness of homomorphisms. However, it does carry connected reductive groups (and even pseudo-reductive groups) to pseudo-reductive groups. The most basic examples of pseudo-reductive groups are products $\prod_i \mathrm{R}_{k'_i/k}(G'_i)$ for a finite (non-empty) collection of finite extensions k'_i/k and absolutely simple and simply connected semisimple k'_i-groups G'_i; the simply connectedness of G'_i ensures that the k-groups $\mathrm{R}_{k'_i/k}(G'_i)$ are perfect, and such Weil restrictions are non-reductive whenever k'_i/k is not separable.

Cartan k-subgroups in pseudo-reductive groups are always commutative and pseudo-reductive. For example, if k'/k is a finite extension and G' is a nontrivial connected reductive k'-group given with a maximal k'-torus T' then $\mathrm{R}_{k'/k}(T')$ is a Cartan k-subgroup of $\mathrm{R}_{k'/k}(G')$ but it is never a k-torus if k'/k is not separable. It seems to be an impossible task to describe general commutative pseudo-reductive groups over imperfect fields, as we will illustrate with numerous examples. (That is, we do not expect there to be any description as convenient as the use of Galois lattices to describe tori.) The representation theory of connected reductive groups rests on the commutative case (tori), so it seems difficult to say anything interesting about the representation theory of general pseudo-reductive groups. The structure of the commutative objects turns out to be the only mystery, in the following sense. We introduce a class of non-commutative pseudo-reductive groups called *standard*, built as certain central quotients

$$\Big(C \ltimes \prod_i \mathrm{R}_{k'_i/k}(G'_i)\Big) \Big/ \prod_i \mathrm{R}_{k'_i/k}(T'_i)$$

with a commutative pseudo-reductive k-group C, absolutely simple and simply connected k'_i-groups G'_i for finite extensions k'_i/k, and a maximal k'_i-torus T'_i in G'_i for each i. (This also recovers the commutative case by taking the collection $\{k'_i\}$ to be empty.)

We give a canonical way to choose such data in this "standard" construction, and prove that if $\mathrm{char}(k) \neq 2, 3$ then *every* non-commutative pseudo-reductive k-group is standard. In this sense, non-commutative pseudo-reductive k-groups away from characteristics 2 and 3 come equipped with an algebraic invariant

$k' = \prod k'_i$ and can be obtained from semisimple groups over its factor fields along with an additional commutative group. One of the main ingredients in our proofs is Tits' theory of root groups, root systems, and pseudo-parabolic subgroups of pseudo-reductive groups; we give a self-contained development because we need scheme-theoretic aspects of the theory.

Building on some non-standard examples of Tits, we introduce a "generalized standard" construction in characteristics 2 and 3 as well as an additional exceptional construction in characteristic 2 (with a non-reduced root system). We prove that this gives a classification theorem in characteristic 3, as well as in characteristic 2 when $[k : k^2] \leqslant 2$. (Many of our proofs are characteristic-free, and in characteristic 2 we get quite far before we need to assume $[k : k^2] \leqslant 2$.) The general classification problem in characteristic 2 without restriction on $[k : k^2]$ is solved in [CP].

Our description of pseudo-reductive groups in terms of connected semisimple groups and commutative pseudo-reductive groups isolates the mysteries of the commutative case in a useful manner. For example, we discuss automorphisms of pseudo-reductive groups and we gain new insight into the large dimensions of root groups and the non-completeness of coset spaces G/P for non-reductive pseudo-reductive k-groups G and pseudo-parabolic k-subgroups P that are "not too big": the root groups are canonically identified with Weil restrictions of $\mathbf{G}_{\rm a}$ from finite extensions of k (apart from some exceptions when $\mathrm{char}(k) = 2$) and the coset spaces G/P are $\prod_i \mathrm{R}_{k'_i/k}(G'_i/P'_i)$ for semisimple G'_i and parabolic k'_i-subgroups P'_i for finite extensions k'_i/k (apart from some exceptional situations in characteristics 2 and 3). As an indication that the difficulties with characteristic 2 are not just a matter of technique, we mention another application of our structure theorem: if $\mathrm{char}(k) \neq 2$ then a pseudo-reductive k-group is reductive if and only if its Cartan k-subgroups are tori, and this is *false* over every imperfect field of characteristic 2.

In one of the appendices we provide the first published account with proofs of Tits' important work on the structure theory of unipotent groups over arbitrary fields of nonzero characteristic. This is an essential tool in our work and we expect it to be of general interest to those who work with algebraic groups over imperfect fields. In another appendix we use our results on root groups (valid in all characteristics, via characteristic-free proofs) to give proofs of theorems of Borel and Tits for arbitrary connected linear algebraic groups G over any field k. This includes $G(k)$-conjugacy of: maximal k-split k-tori, maximal k-split smooth connected unipotent k-subgroups, and minimal pseudo-parabolic k-subgroups. Another such result is the Bruhat decomposition for $G(k)$ (relative to a minimal pseudo-parabolic

k-subgroup of G). In Chapter 3 we prove the existence of Levi k-subgroups of pseudo-reductive k-groups in the "pseudo-split" case.

The most interesting applications of our structure theorem are to be found in number theory, so let us say a few words about this. In [Con2], this structure theorem is the key to proving finiteness results for general linear algebraic groups over local and global function fields, previously known only under hypotheses related to reductivity or solvability. Among the main results proved in [Con2] are that class numbers, Tate–Shafarevich sets, and Tamagawa numbers of arbitrary linear algebraic groups over global function fields are finite. We hope that the reader finds such arithmetic applications to be as much a source of motivation for learning about pseudo-reductive groups as we did in discovering and proving the results presented in this monograph.

Plan of the monograph

We assume the reader has prior familiarity with basics of the theory of connected reductive groups over a field (as in [Bo2]), but we do not assume previous familiarity with earlier work on pseudo-reductive groups. We develop everything we need concerning pseudo-reductivity.

In Part I we establish some basic results that only require the definition of pseudo-reductivity and we carry out many fundamental constructions. We begin in Chapter 1 with an introduction to the general theory, some instructive examples, and the construction of "standard" pseudo-reductive groups. In Chapter 2 we define (following Tits) pseudo-parabolic subgroups and root groups in pseudo-reductive groups over any field k (assuming $k = k_s$ when it suffices for our needs). In Chapter 3 we give the properties of pseudo-parabolic subgroups and root groups, and define the root datum associated to a pseudo-reductive group. We also establish the Bruhat decomposition and existence of Levi subgroups for pseudo-reductive k-groups when $k = k_s$ (to be used in our work in characteristic 2).

Part II takes up the finer structure theory of standard pseudo-reductive groups (in arbitrary characteristic). In Chapter 4, we rigidify the choice of data in the "standard" construction. In Chapter 5 we state our main theorem in general and record some useful results on splitting of certain central extensions. We also prove some invariance properties of standardness, including that standardness is insensitive to replacing the base field with a separable closure, and we reduce the task of proving standardness for all pseudo-reductive groups (over a fixed separably closed base field k) to the special case of pseudo-reductive k-groups G such that $G = \mathscr{D}(G)$ and the maximal geometric semisimple quotient $G^{ss}_{\overline{k}} = G_{\overline{k}}/\mathscr{R}(G_{\overline{k}})$ is a simple $\overline{k}$-group; such G are called (absolutely) *pseudo-simple*. For such G we establish a standardness criterion. This is used in Chapter 6 to prove the general classification away from characteristics 2 and 3, building on the rank-1 case that we handle using calculations with SL_2.

Part III introduces new ideas needed for small characteristics. In Chapter 7 we construct a class of pseudo-reductive groups (discovered by Tits) called *exotic*, which only exist over imperfect fields of characteristics 2 and 3. In Chapter 8 we study the properties of these groups, including their automorphisms and the splitting of certain central extensions. The structure theory in characteristic 2 involves new difficulties and requires additional group-theoretic constructions (known to Tits) resting on birational group laws. The absolutely pseudo-simple case is studied in Chapter 9, and the classification in characteristics 2 and 3 is completed in Chapter 10 using a "generalized standard" construction. In characteristic 2 we need to assume k is almost perfect (i.e., $[k : k^2] \leqslant 2$) in order to prove that the constructions we give are exhaustive. (If $[k : k^2] > 2$ then there are more possibilities, and these are classified in [CP].) Chapter 11 gives applications of our main results.

Part IV contains the appendices. Appendix A summarizes useful (but largely technical) facts from the theory of linear algebraic groups, with complete proofs or references for proofs. We frequently refer to this appendix in the main text, so the reader may wish to skim some of its results and examples (especially concerning Weil restriction) early in the process of learning about pseudo-reductive groups. Appendix B proves unpublished results of Tits [Ti1] on the structure of smooth connected unipotent groups over arbitrary (especially imperfect) fields with nonzero characteristic. This is crucial in many proofs of the deeper aspects of the structure theory of pseudo-reductive groups. Appendix C proves many interesting rational conjugacy theorems, and the Bruhat decomposition, announced by Borel and Tits in [BoTi2] for all smooth connected affine groups over arbitrary fields, and discusses root data, BN-pairs, and simplicity results in such cases too.

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Terminology, conventions, and notation

Algebraic geometry. All rings are understood to be commutative with identity unless we indicate otherwise. For example, if B is such a ring then a B-algebra is understood to be commutative with the "same" multiplicative identity element. For a field k, we write k_s to denote a separable closure and $\overline{k}$ to denote an algebraic closure. If X is a scheme over a ring k and R is a k-algebra then $X_R = X \otimes_k R$ denotes the R-scheme obtained by scalar extension. We use similar notation $M_R = M \otimes_k R$ for k-modules M.

We often work with schemes and maps between them by thinking functorially. In particular, if k is a ring and X is a k-scheme then a *point* of X often means an element in $X(R)$ for a k-algebra R that we allow to vary. The context should always make clear if we instead mean points of the underlying topological space of a scheme (such as generic points). For a map of schemes $f : Y' \to Y$ and a locally closed subscheme $X \hookrightarrow Y$ we write $f^{-1}(X)$ to denote the scheme-theoretic pullback $X \times_Y Y'$ viewed as a locally closed subscheme of Y'.

Preimages and intersections of subschemes are always taken in the scheme-theoretic sense (and so may be non-reduced) unless we indicate otherwise. The underlying reduced scheme of a scheme X is denoted $X_{\rm red}$. Beware that if k is an imperfect field and G is an affine k-group scheme of finite type then $G_{\rm red}$ can fail to be a k-subgroup scheme of G! Moreover, even when it is a k-subgroup scheme, it may not be k-smooth. See Example A.8.3 for natural examples. (A case with good behavior over any field k is that if T is a k-torus and M is a closed k-subgroup scheme of T then $M_{\rm red}$ is a smooth k-subgroup; see Corollary A.8.2.)

An extension of fields K/k is *separable* if $K \otimes_k \overline{k}$ is reduced. Note that K/k is not required to be algebraic (e.g., we can have $K = k(x)$, or $K = k_v$ for a global field k and place v of k).

In algebraic geometry over a field k, Galois descent is a useful procedure for descending constructions and results from k_s down to k. We sometimes need to descend through an inseparable field extension. For such purposes, we use faithfully flat descent. The reader is referred to [BLR, 6.1] for an introduction to that theory, covering (more than) everything we need.

The reader is referred to [BLR, Ch. 2] for an excellent exposition of the theory of *smooth* and *étale* morphisms of schemes. (For example, a map of schemes $X \to Y$ is smooth if, Zariski-locally on X, it factors as an étale morphism to an affine space over Y.) We sometimes need to use the *functorial criterion* for a morphism to be smooth or étale, so we now recall the statement of this criterion in a special case that is sufficient for our needs. If X is a scheme locally of finite type over a noetherian ring R then $X \to \operatorname{Spec} R$ is smooth (resp. étale) if and only if $X(R') \to X(R'/J')$ is surjective (resp. bijective) for every R-algebra R' and ideal J' in R' such that ${J'}^2 = 0$.

Algebraic groups. For an integer $n \geqslant 1$, a commutative group scheme is *n-torsion* if it is killed by n. We write $C[n]$ to denote the n-torsion in a commutative group scheme C (with $n \geqslant 1$), and $\mu_n = \mathrm{GL}_1[n]$. We also write $[n]$ to denote multiplication by n on a commutative group scheme.

For a smooth group G of finite type over a field k, its *derived group* is $\mathscr{D}(G)$ (Definition A.1.14).

We always use the notions of *kernel*, *center*, *centralizer*, *central*, *normalizer*, *subgroup*, *normal*, and *quotient* in the scheme-theoretic sense, though sometimes we append the adjective "scheme-theoretic" for emphasis; see Definitions A.1.8–A.1.11.

If S is a scheme (generally affine) and G is an S-group scheme then $\underline{\mathrm{Aut}}(G)$ denotes the functor on S-schemes assigning to any S' the group $\mathrm{Aut}_{S'}(G_{S'})$ of S'-group automorphisms of $G_{S'}$. We likewise define the functors $\underline{\mathrm{End}}(G)$ when G is commutative and $\underline{\mathrm{Hom}}(G,H)$ for commutative G and H. If we wish to consider functors respecting additional structure (such as a linear structure on a vector group; see Definition A.1.1) then we will say so explicitly.

For a smooth affine group G over a field k and a k-split k-torus $T \subseteq G$, $\Phi(G,T)$ denotes the set of *nontrivial* weights of T under the adjoint action of G on $\mathrm{Lie}(G)$ (i.e., the set of nontrivial k-homomorphisms $a : T \to \mathbf{G}_{\mathrm{m}}$ such that the a-weight space in $\mathrm{Lie}(G)$ is nonzero). This definition also makes sense for any group scheme G locally of finite type over k, but we only require the smooth affine case. (By using Lemma A.8.8, this definition carries over to smooth group schemes and split tori over any connected non-empty base scheme, but we will not need this generalization.)