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# Part I

## First Notions

 

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# 1 Introduction

In their great variety, from contests of global significance such as a championship match or the election of a president down to a coin flip or a show of hands, games and elections share one common feature: each game or election offers the possibility of a final, decisive result obtained according to well-established rules, a public **outcome**. Mathematics offers a similar possibility. In mathematics, the rules are founded in the laws of logic and represent a formalization of our basic common sense. An outcome in mathematics, to pursue the analogy, is a **theorem**, a statement that can be proven true. The outcome of a game or election may be surprising or expected. Similarly, a theorem can either defy intuition or confirm a well-evidenced conjecture. Just as a game or an election separates winners from losers, a proven theorem distinguishes true statements from false ones, creating a new fact-of-the-matter about mathematics.

This book is an introduction to the mathematical theory of games and elections. We pursue the analogy between analyzing a game or election and developing a mathematical theory somewhat further before turning to our main topics. First, just as a game or an election creates a new language of specialized terminology, a mathematical theory begins with the formulation of *definitions*. A mathematical **definition** is a precise and verifiable description of an object of study. We adopt the following convention for definitions: when we define a new term, we use **boldface**. Thus the terms "outcome," "theorem," and "definition" were defined earlier. (Of course, subsequent definitions will be more technical than these.) We use *italics* to indicate we are mentioning a term that has not been defined yet but will be defined later. For example, we mentioned the term "definition" before subsequently defining it earlier.

The best way to learn to play a new game is often to just give it a try. In mathematics, the corresponding hands-on method of learning is the study of *examples*. By an **example**, we mean a specific instance of a definition or, alternately, a particular consequence of a theorem. A great advantage of our chosen topics is the wealth of examples. Not only do we have many familiar games and election methods available, creating new examples is as easy as making up rules. When referring to games or election methods, we will use capital letters.

We organize our study around the roles of *chance, strategy, and choice*. We define these terms loosely here. By an act of **chance**, we mean an event whose outcome is not predetermined or controlled by human agency. By an act of **strategy**, we mean a situation in which a person chooses among possible moves. Finally, by a **social choice**, we mean a decision by a group of individuals (the society) to select one of several competing alternatives. We briefly discuss the history of the study of these topics. As we do so, we highlight three questions arising from representative examples in each area. These questions will motivate more advanced developments in the second part of the book.

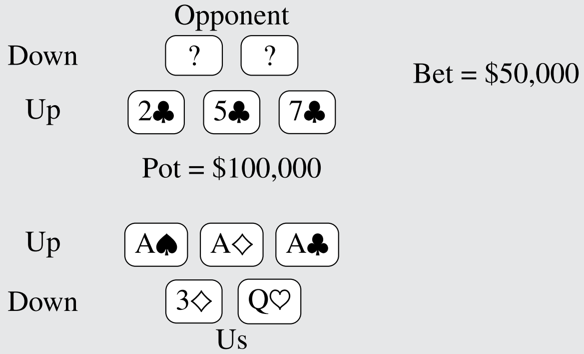
The study of pure chance belongs to the field of **probability theory**, the science of odds. The modern theory has roots in gambling games in Paris casinos. In the 1650s, a correspondence between two luminaries of the time, Blaise Pascal and Pierre de Fermat, led to a solution of a basic problem on iterated rolls of the dice, laying the groundwork for the modern mathematical theory. The twentieth century saw deep entrenching of the probabilistic viewpoint with the discovery of quantum mechanics and the development of mathematical statistics. Probability theory is now part of the common vernacular as we talk about sports, the stock market, and the weather.

Among the simplest versions of Poker is the game called **Five-Card Stud**. Five-Card Stud can be played between two or among several players. Each player is first dealt two **down** cards (cards not revealed to other players). The remaining three cards for each player are dealt **up** (revealed to all players) one by one. After each player receives a new up card, the players bet on their evolving hands. When a player makes a bet, each subsequent player may either **fold** (resign and give up any chance at the pot), **call** (match the bet), or **raise** (match the bet and make a new, added bet). When all the cards are turned and all players have either folded or called the last bet, the eligible players show their down cards. The player with the best Poker hand takes the pot. Here is an example.

**Example 1.1 A Hand of Five-Card Stud**

We consider the following scenario in which only two players survive to the final round of betting. The cards are as shown. The pot is $100,000. Our opponent has bet $50,000 more. We are faced with two choices: call the $50,000 bet or fold and lose our chance at the pot.

**A Hand of Five-Card Stud**



We will introduce the rankings of Poker hands in Chapter 14. Here we simply observe that our hand is **Three of a Kind** (three cards with the same value, in this case aces). The only possible hand our opponent can have that beats ours is a **Flush**, five cards of the same suit. Specifically, our opponent must be holding two clubs to beat us.

Poker is a game involving both chance and strategy. The strategy in Five-Card Stud arises solely in the betting, the decision to call or fold or how much to bet to open and whether to raise. To keep this particular example in the realm of pure chance, we assume that we know that our opponent bets on a losing hand only 3 percent of the time. A bet on a losing hand made with the intention of inducing an opponent to fold is called a **bluff**. Our first question is the following:

**Question 1.2** *Should we call or fold in A Hand of Five-Card Stud?*

If we call, we risk $50,000 more but have a chance at what would be a $200,000 pot! If we fold, the game is over. Of course, the decision to call or fold in a game of Poker is a personal and financial one. We will propose an answer to Question 1.2 in the form of an *expected value* in Chapter 14. The analysis of A Hand of Five-Card Stud will lead us to discover a powerful method in probability theory called *Bayes' Law*.

Poker was a central example in the development of **game theory**, the study of games of strategy and strategic conflict. Émile Borel began the mathematical study of strategy in the 1920s. His work included an analysis of a simplified version of Poker. Borel conjectured the existence of an *equilibrium* for games in which the players are in total conflict. In 1928, John von Neumann proved Borel's conjecture true in what is now known as the *Fundamental Theorem of Game Theory*. We will prove a special case of this theorem in Chapter 15.

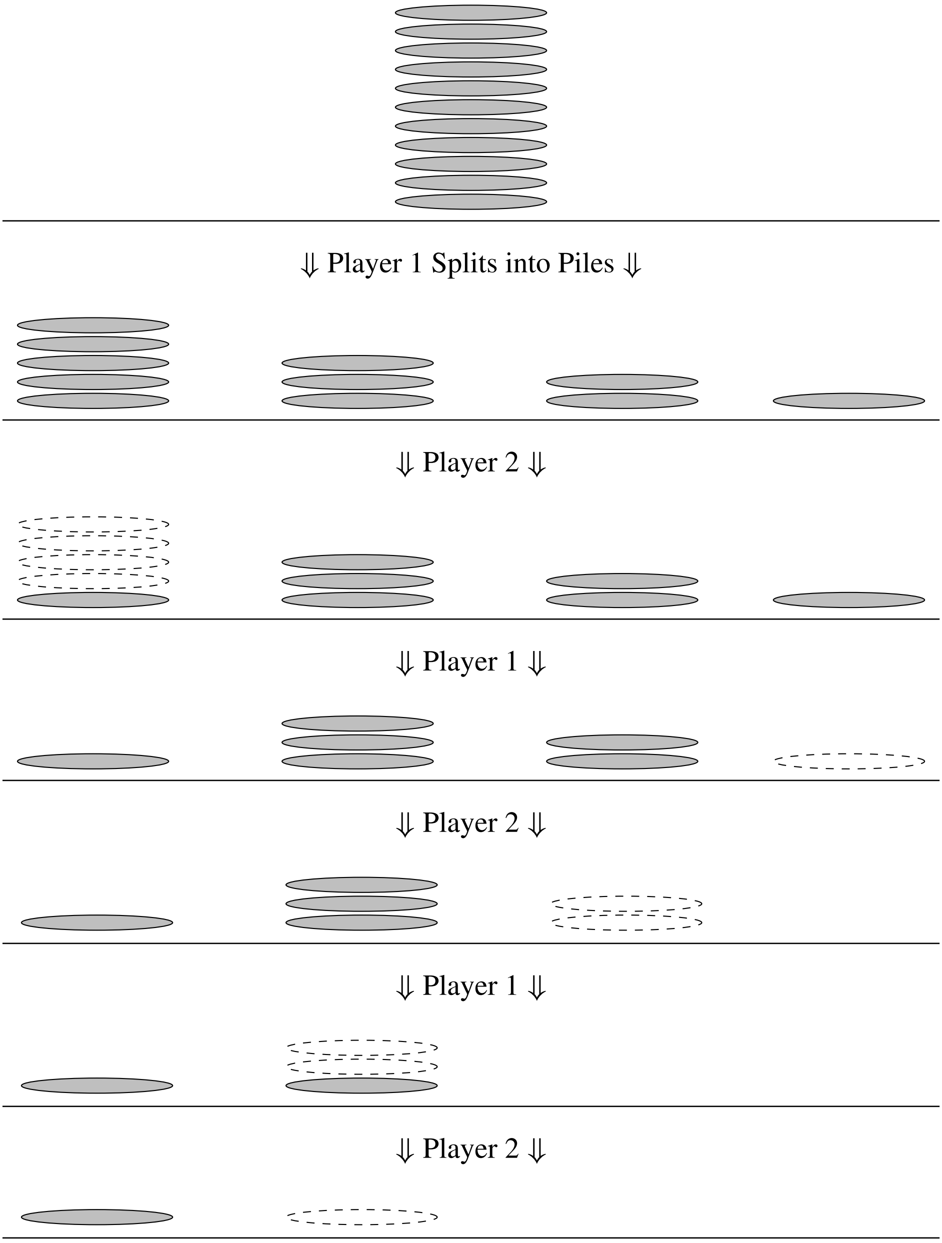
John Nash analyzed a three-player Poker game in his seminal 1950s Ph.D. thesis. The analysis was a first application of his spectacular generalization of von Neumann's Theorem. The celebrated Nash Equilibrium Theorem is now the centerpiece of modern game theory, with applications in fields ranging from economics to business to evolutionary biology. Nash was awarded the Nobel Prize in Economics in 1994 for his work.

Perhaps the most purely mathematical games are those with no role for chance. Games such as Checkers and Chess present contests of pure strategy and high complexity, rewarding superior cleverness and the ability to foresee future contingencies. We introduce a simple game of this class that can be played with just a handful of Poker chips, 11 chips in this case. Our example is a variation on the game called *Nim*.

**Example 1.3 A Game of Nim**

The game we propose is played as follows: the first player, referred to as Player 1, is handed 11 Poker chips. She divides the chips into any number of piles. The second player, Player 2, then takes any number of chips (but at least one) from one of the piles. Player 2 can only take chips from one pile but can take as many as desired, including the whole pile. Player 1 goes next, taking any number of chips (but at least one) from one of the remaining piles. The game continues in this way, with each player taking chips from one pile on his or her turn. The game is over when all the chips are taken. The last player to take chips is the winner. Here is a sample round:

A Game of Nim




Player 1 Wins

Of course, the preceeding scenario is only one of the many ways the game can be played. We pose the following question:

**Question 1.4** *In A Game of Nim, which player has the advantage, Player 1 or Player 2?*

We address Question 1.4 in Chapter 17. The answer comes as a consequence of a remarkable theorem published by Charles Bouton in 1901. Bouton gave a complete mathematical solution to the game of Nim, setting the stage for extensive twentieth-century research on the mathematics of games of pure strategy.

The study of elections, or **social choice theory**, is an interdisciplinary field with branches in political science and philosophy as well as mathematics. The question of how to conduct elections is one faced by every political entity. The answers often reveal the character of the government. Even among democracies, the ideals of fairness and universal representation are pursued in various ways. Here we may compare, for example, the majority runoff methods for presidential elections of Ireland and France with the method of delegates used in the U.S. Electoral College.

The mathematical theory of elections traces back to a debate carried out in academic journals in late eighteenth-century France. The Chevalier Jean-Charles de Borda introduced the rank-ordering method now bearing his name for elections involving multiple candidates. Borda argued for the suitability of his method by introducing what is now known as a *fairness criterion*. During this same period, the Marquis Nicolas de Condorcet uncovered a basic obstacle to fairness for elections called the *Condorcet Paradox*. Condorcet critiqued Borda's method for ignoring the majority opinion in certain cases. He argued for a voting method featuring head-to-head elections. Condorcet based his arguments for his majoritarian voting method on his proof of the Jury Theorem. We prove a version of this theorem in Chapter 21.

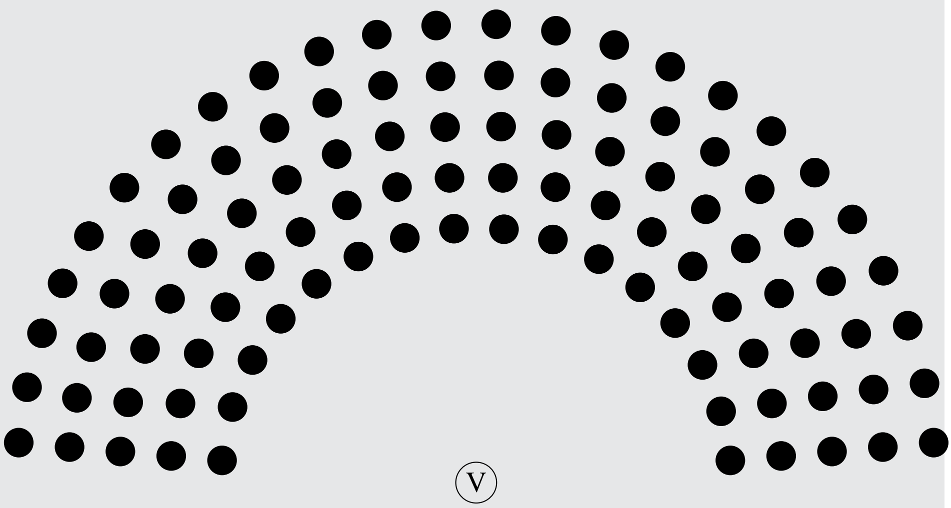
The foundation for the modern theory of social choice is the 1950s' doctoral thesis of Kenneth Arrow. Arrow developed a mathematical language for the field, giving precise definitions for election mechanisms and their fairness. Within this framework, he proved his celebrated Impossibility Theorem, establishing that all democratic election mechanisms allowing more than two candidates violate particular principles of fairness. His work opened the door to wide-ranging research on the mathematics of voting and elections. Arrow was awarded the Nobel Prize in 1972. We prove Arrow's Impossibility Theorem in Chapter 23.

Social choice theory offers a wealth of problems concerning the distribution of power in legislative systems. We pose a simple question about one of the most elegant legislative bodies, the U.S. Senate.

**Example 1.5 The U.S. Senate**

The U.S. Senate is comprised of 100 senators: two senators for each of the 50 states. A bill passes the Senate with a majority vote. However, in the event of a 50–50 tie, the vice president, who is known as the President of the Senate, casts the tie-breaking vote.

The United States Senate




The question we pose here is the following:

**Question 1.6** *Who has more power in the U.S. Senate, an individual senator or the vice president?*

We note that the U.S. Senate presents many fascinating problems of voting strategy, with roll-call votes, amendment riders, and filibusters available among the various methods. Our question concerns only the basic power distribution of the voting system. We answer Question 1.6 in Chapter 13 using the idea of a *power index*, a notion that connects the study of elections with probability theory. We will see that the answer to Question 1.6 is essentially the proof of a famous *combinatorial identity*.

We conclude with some general remarks about the text. As the preceding discussion suggests, we emphasize the interplay between the ideas of chance, strategy, and choice in our study of these three topics. The integration will be especially apparent in the beginning chapters. As we delve deeper into the separate fields of probability, game, and social choice theory, we will focus more closely on the area at hand.

In addition to the topics of games and elections, the text is intended to give an introduction to mathematics. As discussed earlier, we will begin with precise definitions and make use of examples to illustrate ideas and build intuition. Although our approach will be elementary, we will move steadily in our analysis so that by the end of the text we will be in a position to prove some important theorems about games and elections.

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# 2 Games and Elections

In this chapter we formulate mathematical definitions of a *game* and an *election* and give examples. The examples of games will indicate some of the categories we will study and will also serve to introduce the two main devices we will use to represent games: the *game tree* and the *payoff matrix*. For elections, we introduce the notions of *true preferences* and *insincere voting.* We consider the famous example of the 2000 presidential election in the state of Florida as an illustration of the connection between games and elections.

We begin by thinking a bit about how to define the notion of a *game*. The term has broad meanings. What properties should we use to limit the scope? We can probably agree that a game should involve an act of competition with roles for skill, ingenuity, and maybe luck. Should a game be fun? Should it be fair? Fun is, of course, a matter of taste. Fairness also seems difficult to ensure. A grandmaster can play a game of Chess against a novice, but surely this this is not a "fair" game. However, allowing a weaker player to have a handicap destroys the intrinsic fairness of a game but may be the only way to create a good match.

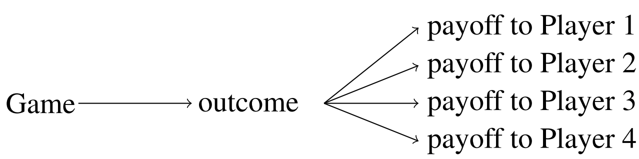
We define a game in a manner that is quite inclusive, making no commitment either way as to the questions of fun, fairness, or other such qualities. We focus instead on what a game produces. This is the **outcome**, by which we mean some particular, well-defined final status such as "win," "loss," or "tie." Whatever the method or means of play, when the dust settles and the game is over, we expect to have a well-defined final result.

Further, we often want to know the score of the game or the winnings for each player. We will refer to these as the *payoffs*. Precisely, the **payoffs** are numeric quantities assigned to each player for each particular outcome. We can think of the payoff to a player as either points or a dollar amount. Payoffs will be either zero (no gain), positive (a gain), or negative (a loss). When appropriate, we will express payoffs as dollar amounts. We can now formulate a mathematical definition of a game.

**Definition 2.1** A **game** is an event or mechanism involving two or more players that produces a well-defined outcome resulting in a payoff to each player.

We can picture the playing of a game, here, a four-player game, as a process:

**A Four-Player Game**




We often focus on two-player games. In this setting, we have the special class of **win-lose games,** two-player games in which the only outcomes are win and lose. If ties are also possible, then we say this is a **win-lose-tie game.** The payoffs for such games may be taken, implicitly, to be win $= +1$, lose $= -1$, and tie $= 0$.

We can classify games according to the method by which the outcome is achieved. By a **game of pure chance** we mean a game in which the outcome is produced by an act of chance. Examples include the children's game Chutes and Ladders and the casino game Roulette. A **game of pure strategy** is a game in which there is no role for chance. The strategic decisions of the players lead directly to the outcome of the game. Examples here include Tic-Tac-Toe, Checkers, and Chess. Perhaps the simplest example of a game is the following:

**Example 2.2 Heads or Tails**

This is the classic method of making a decision between two competing options. One player calls "Heads" or "Tails." The other player flips a coin. The calling player wins if he or she calls the flip correctly. Otherwise, the flipping player wins.

What kind of game is Heads or Tails? Note that the outcome of Heads or Tails is achieved in two steps. The first step in the game involves a strategic decision, albeit a rather trivial one. The second step is an act of chance. We refer to games involving both chance and strategy as **games mixing chance and strategy**. These include many of the most entertaining games, such as Monopoly, Scrabble, Poker, and Bridge.

The game Heads or Tails serves to introduce an important device called the *game tree*. A game tree is used to represent games in which players take turns, one by one, until the final outcome is produced. Such a game is called a **sequential-move game**. Most board games are sequential-move games. The game tree shows the progress of such a game as it moves turn by turn to the outcome. The last stage is the outcome of the game and so, in our case, produces a winner and a loser. We label accordingly: