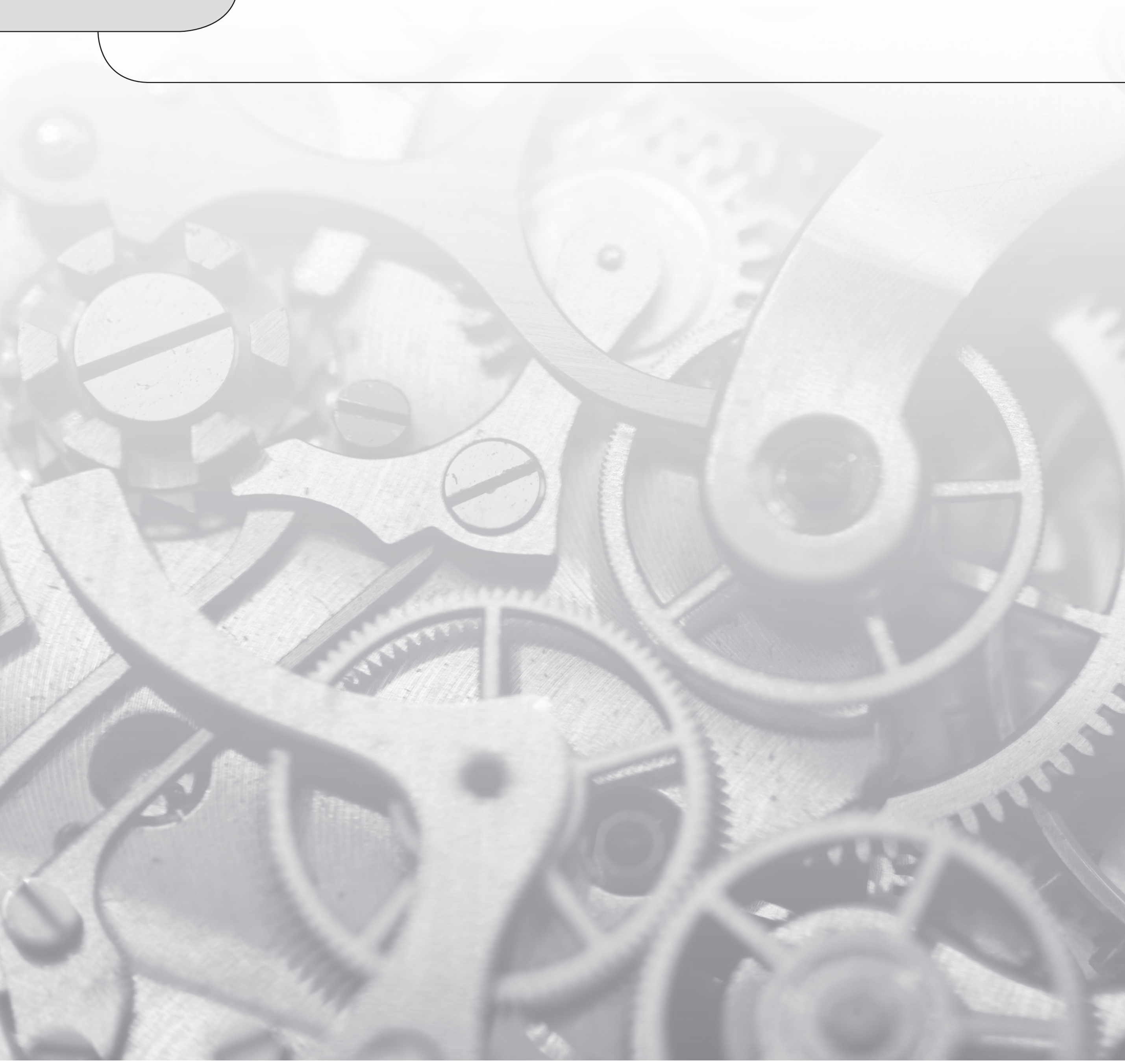
1 Difference equations



Cambridge University Press
978-1-107-08329-5 - Dynamic Economic Analysis: Deterministic Models in Discrete Time
Gerhard Sorger
Excerpt
More information

1 Basic concepts

Difference equations are formal models of dynamical systems in which time is assumed to evolve in discrete periods. Models of this kind are used in many areas of economic research, including macroeconomics, monetary economics, resource economics, game theory, etc. In this chapter, we introduce some basic concepts and illustrate them by a number of selected examples. Throughout the book, we restrict the presentation to deterministic systems, that is, we do not consider any models involving uncertainty.

One of the simplest types of difference equations arises through repeated iterations of one-dimensional maps. Because of their conceptual simplicity, we start our discussion of difference equations with these models, proceeding rather informally and without proving any theorems. Later in chapter 4, we shall see that even these simple difference equations can generate surprisingly rich dynamics. In section 1.2, we continue by introducing a general class of explicit difference equations and by extending the basic concepts to this framework. We also increase the level of rigor and formally prove several properties of the solutions of explicit difference equations including their existence and their uniqueness for a complete set of initial or boundary conditions. Finally, in section 1.3, we argue that economic models often take the form of implicit difference equations. Unfortunately, neither the existence nor the uniqueness of solutions to such equations can be ensured, a fact that we illustrate by means of a detailed economic example.

1.1 One-dimensional maps

Suppose that the economic system under consideration can be described by a single variable $x \in X$, where $X \subseteq \mathbb{R}$ is a non-empty interval on the real line. Depending on the context, the variable x can measure the productive capital available in the economy, the price of a commodity, the stock of a resource, the fraction of the population with a certain characteristic, etc. We shall refer to x as the *system variable* and to X as the *system domain*. The system domain contains all possible values of the system variable. Suppose furthermore that the economic system is a dynamic one, that is, that

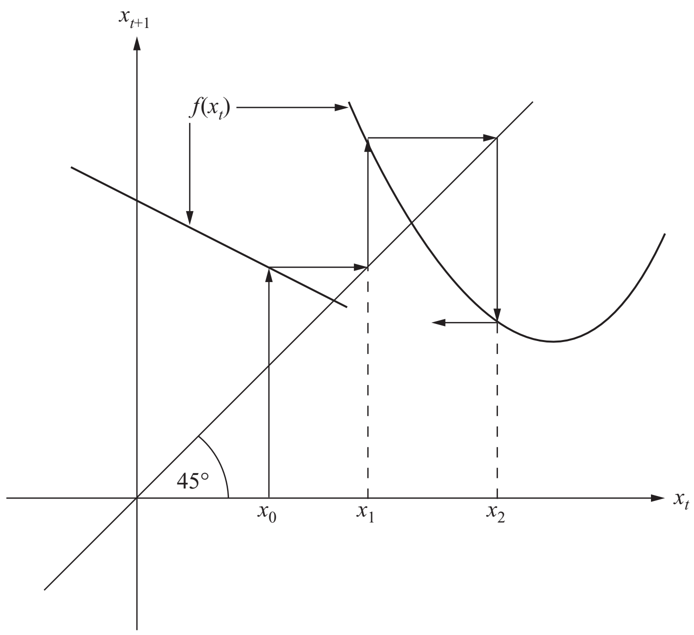


Figure 1.1 Construction of a trajectory

the value of x may change over the course of time. We assume that time is measured in discrete periods $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and that the variable x can change its value at most once in every period. The set $\mathbb{N}_0$ of all periods is called the *time domain* or *time horizon*. The interpretation of time periods is also dependent on the context. For example, the variable t may refer to calendar time (days, quarters, years, etc.) or to notional time (decision periods, applications of an operator, generations, etc.). Throughout this book, we indicate the time period by a subscript. In particular, x_t is the value of the system variable in period t. The dynamic evolution of the system is described by a function $f : X \mapsto X$, which is often called the *law of motion* or the *system dynamics*. More specifically, if the system variable has the value x_t in period $t \in \mathbb{N}_0$, then it has the value

$$x_{t+1} = f(x_t) \tag{1.1}$$

in period $t + 1$.

Equation (1.1) is an example of a *difference equation*, because it relates the values of the system variable in different time periods to each other. A solution to this equation is a sequence $(x_t)_{t=0}^{+\infty}$ such that $x_t \in X$ and (1.1) hold for all $t \in \mathbb{N}_0$. Such a sequence is referred to as a *trajectory* of (1.1).

There exists an extremely simple but powerful method for analyzing the dynamics generated by equation (1.1). To apply this method, we must draw the graph of the function f as well as the 45° line into a (x_t, x_{t+1})-diagram as shown in figure 1.1.

Suppose that the system has initially the value x_0. We start by locating the point $(x_0, 0)$ on the horizontal axis and by drawing a vertical line from this point to the graph of f. Because of (1.1), it follows that the intersection of this vertical line with the graph of f occurs at the point (x_0, x_1). Starting at that point, we then draw a horizontal line to the 45° line to get to the point (x_1, x_1). We can repeat the same construction over and over again, that is, starting from the most recently determined system value we first draw a vertical line to the graph of the function f and then a horizontal line to the 45° line. In this way, we can construct as many elements of a trajectory of the difference equation (1.1) as we wish. It should be emphasized that this technique does not require any continuity properties of the system dynamics f.

In the following example, the graphical method is applied to a famous model of economic growth.

Example 1.1 The Solow–Swan model is an aggregative model of economic growth. Suppose that the aggregate capital stock available in the economy at the beginning of period t is K_t and that the size of the labour force in period t is L_t. The labour force is assumed to grow at the exogenously given rate $n > -1$ such that

$$L_{t+1} = (1+n)L_t$$

holds for all $t \in \mathbb{N}_0$. Using its factor endowments with capital and labour, K_t and L_t, respectively, the economy is able to produce the amount $F(K_t, L_t)$ of output in period t, where $F : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ is the aggregate production function. This output can be used for consumption or it can be invested to form new capital. One of the central assumptions of the Solow–Swan model is that in every period the same fraction of output is invested. Denoting this fraction by $s \in [0, 1]$, it follows that aggregate investment in period t is equal to $sF(K_t, L_t)$, whereas aggregate consumption in period t is $(1-s)F(K_t, L_t)$. By the elementary rules of bookkeeping, it follows that the aggregate capital stock at the beginning of period $t+1$ is equal to the aggregate capital stock that was available at the beginning of period t, plus investment during period t and minus depreciated capital. Assuming finally that a constant fraction $\delta \in [0, 1]$ of existing capital depreciates in every period, it follows that

$$K_{t+1} = K_t + sF(K_t, L_t) - \delta K_t.$$

This equation is not of the form (1.1) as it contains both the capital stock and the labour force. However, if we assume that the production technology exhibits constant returns to scale, that is, that the function F is homogeneous of degree 1, then we can rewrite the above equation in the form of (1.1). The function F is homogeneous of degree 1 if and only if $F(\lambda K, \lambda L) = \lambda F(K, L)$ holds for all triples of non-negative numbers (K, L, λ). Dividing the displayed equation from above by $(1+n)L_t$ and using

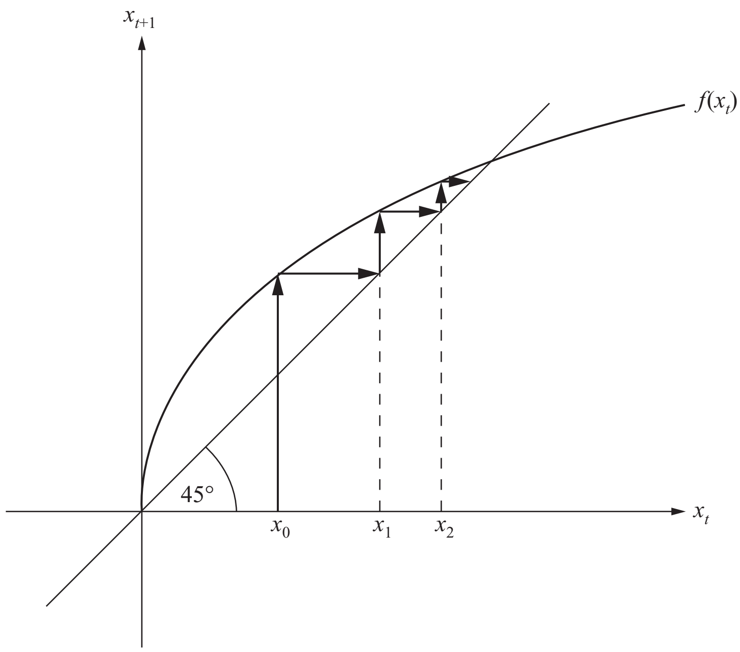


Figure 1.2 Construction of a trajectory of the Solow–Swan model

homogeneity of degree 1 of F plus the fact that $(1+n)L_t = L_{t+1}$ holds, it follows that

$$\frac{K_{t+1}}{L_{t+1}} = \frac{K_t/L_t + sF(K_t/L_t, 1) - \delta K_t/L_t}{1+n}.$$

Finally, by introducing the system variable $x_t = K_t/L_t$, which denotes the capital stock per worker, we obtain equation (1.1) with

$$f(x) = \frac{(1-\delta)x + sF(x, 1)}{1+n}.$$

As for the system domain, we may choose $X = \mathbb{R}_+$.

Figure 1.2 shows the graph of the function f in the case where the production function F is given by $F(K, L) = \sqrt{KL}$. The figure also shows the 45° line and illustrates how a trajectory of the Solow–Swan model can be constructed along the lines explained above.

The graphical method also helps to identify the so-called *fixed points* of equation (1.1), that is, those values of the system variable which remain fixed under the law of motion f. These fixed points are simply the intersections of the graph of f and the 45° line. In the example depicted in figure 1.2, we can see two such fixed points. One of them is $x = 0$, whereas the other one is in the interior of the system domain $X = \mathbb{R}_+$. It is straightforward to see from the figure that all trajectories of the Solow–Swan model, except for the one emanating from the initial value $x = 0$, converge monotonically to the interior fixed point.

Before we discuss more general types of difference equations, let us consider two more examples of one-dimensional maps, one from evolutionary game theory and another one without any economic background.

Example 1.2 Consider the doubly symmetric two-player normal form game with payoff matrix

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

where a and b are positive numbers. We denote the strategy corresponding to the first row or column by A and the strategy corresponding to the second row or column by B. The payoff matrix says that the players receive a if both of them choose A, they receive b if both of them play B, and they receive 0 otherwise. Since a and b are positive, it follows that the players are better off if they choose the same strategies than if they choose different ones. The game is therefore called a coordination game.

The above game has two pure strategy Nash equilibria, (A, A) and (B, B), and one mixed strategy Nash equilibrium in which A is chosen with probability $\bar{x} = b/(a+b)$. There is no obvious reason why one of these equilibria should be preferred over the others. The replicator dynamics have been suggested as a selection device. To explain this device, suppose that there exists a continuum of measure 1 of players who are randomly matched in every period to play the game. Denote the fraction of players who choose A in period t by x_t. The expected payoff of an A-player is therefore ax_t and the expected payoff of a B-player is $b(1 - x_t)$. The average payoff in the entire population is given by $ax_t^2 + b(1 - x_t)^2$.

According to the discrete-time replicator dynamics the growth factor of the measure of players who play a certain strategy is given by the ratio of the expected payoff of players choosing that strategy and the average payoff in the entire population, that is

$$\frac{x_{t+1}}{x_t} = \frac{ax_t}{ax_t^2 + b(1 - x_t)^2}.$$

Multiplication by x_t shows that this equation has the form of (1.1) with

$$f(x) = \frac{ax^2}{ax^2 + b(1 - x)^2}.$$

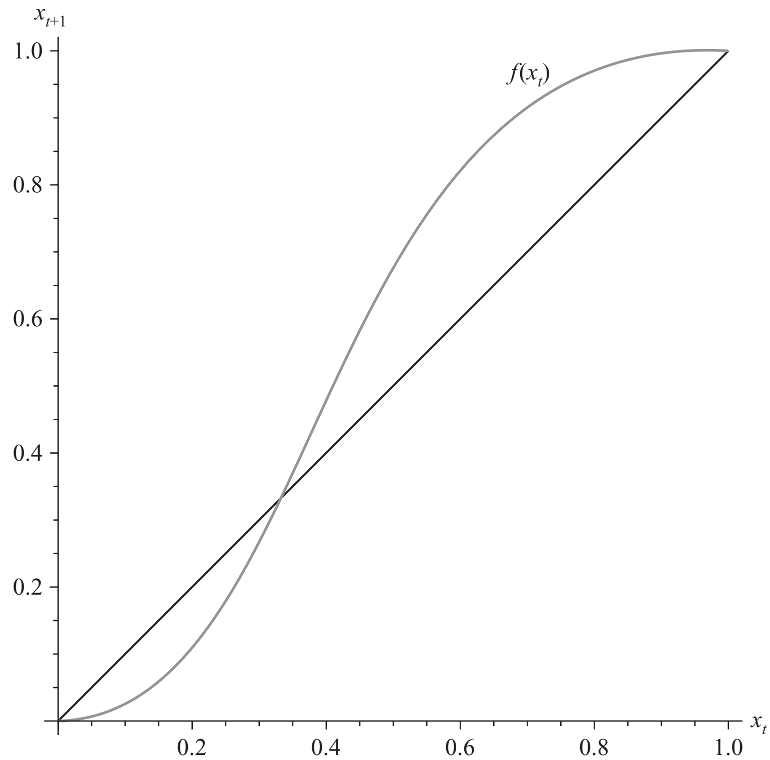


Figure 1.3 The replicator dynamics

If there are no A-players in period t, that is, if $x_t = 0$, then there will be no A-players in the next period either. This is the case because $f(0) = 0$. Analogously, if $x_t = 1$, then there will be no B players in period $t + 1$, because $f(1) = 1$. Thus, the entire population playing the same pure strategy corresponds to a fixed point of the replicator dynamics. There exists, however, a third fixed point, that is, a third solution to the equation $f(x) = x$. This solution is given by $x = \bar{x} = b/(a + b)$. Whenever the population consists of exactly $\bar{x}$ agents who play strategy A and exactly $1 - \bar{x}$ players who choose strategy B, then it follows that this situation will persist forever. Note that the fraction of A-players in this fixed point coincides exactly with the probability of playing A in the mixed strategy Nash equilibrium.

The set of all possible values of the system variable x in this example is the unit interval $X = [0, 1]$. Figure 1.3 displays the graph of the function f along with the 45° line for the case where $a = 2$ and $b = 1$. We can clearly see the three fixed points corresponding to the three Nash equilibria. Graphical analysis along the lines suggested above shows that all trajectories of the replicator dynamics converge to one of these fixed points. More specifically, whenever $x_0 \in (\bar{x}, 1]$, then it follows that the trajectory emanating from x_0 converges to 1, and whenever $x_0 \in [0, \bar{x})$, then it converges to 0. If $a > b$ (as is the case in the example depicted in figure 1.3), then it holds that $1 - \bar{x} > \bar{x}$ such that the set of initial distributions of strategies from which the fixed point corresponding to the Nash equilibrium (A, A) is reached is larger than the set from which the fixed point corresponding to (B, B) is reached. The mixed strategy Nash equilibrium seems to be particularly fragile in the replicator dynamics as it is not reached from any initial distribution that differs from $\bar{x}$. Using the size of the set from which a given fixed point (that corresponds to a Nash equilibrium) is reached as an equilibrium selection criterion therefore leads to the selection of the pure strategy Nash equilibrium (A, A).

The last example in this section, although not originating from any economic application, has strong similarities to the previous example. It shows how an appropriately defined dynamic model can be used to single out one of multiple solutions of a purely static problem. Once again, the graphical method for the analysis of one-dimensional maps turns out to be very useful.

Example 1.3 Let $a > 0$ be given and consider the expression

$$x = a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}, \tag{1.2}$$

that is, the infinitely continued exponentiation of the positive real number a. To compute the value of x for a given value of a, we could exploit the fact that whenever x is given by (1.2) then it must obviously satisfy the equation

$$x = a^x. \tag{1.3}$$

We could therefore try to solve (1.3) instead of (1.2). Unfortunately, this does not necessarily yield a unique answer. For example, if $a = \sqrt{2}$, then both $x = 2$ and $x = 4$ are solutions of (1.3) such that the static model given by equation (1.3) yields an ambiguous answer. Let us therefore try to make the interpretation of (1.2) more precise by using a recursive (dynamic) approach. One possibility to do this is to define x as the limit of the sequence

$$a, a^a, a^{a^a}, \ldots.$$

A more practical way of expressing this is $x = \lim_{t \to +\infty} x_t$, where $x_0 = a$ and $x_{t+1} = f(x_t)$ for all $t \in \mathbb{N}_0$ and where the function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is defined by $f(x) = a^x$. Note that every solution to (1.3) qualifies as a fixed point of the above one-dimensional map. Plotting the graph of the function f for the parameter value $a = \sqrt{2}$ (see figure 1.4) and applying the graphical method introduced earlier in the present section, it is easy to verify that no trajectory emanating from an initial

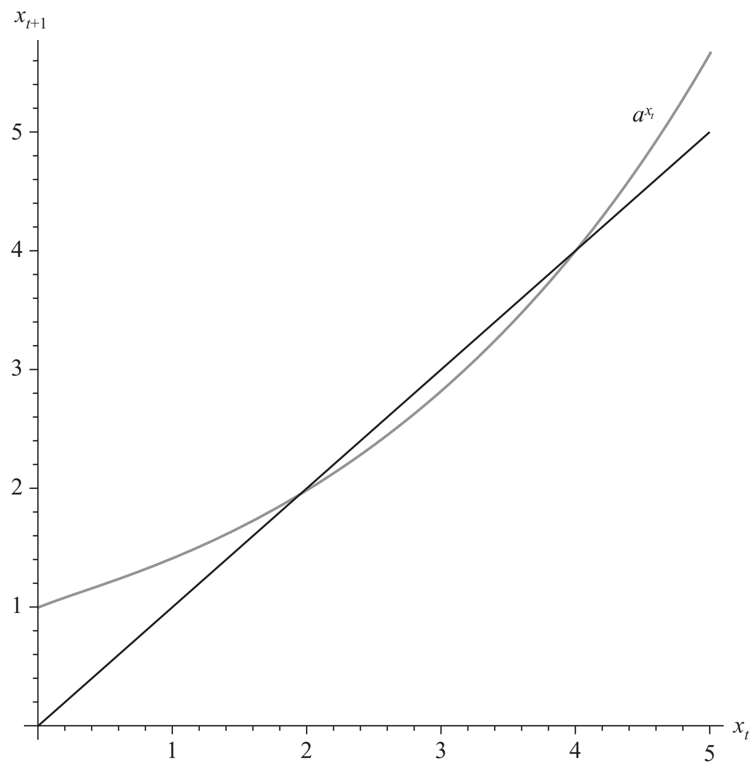


Figure 1.4 The graph of the function a^x for $a = \sqrt{2}$

value $x_0 \neq 4$ can ever approach the fixed point $x = 4$. In particular, the trajectory starting at $x_0 = a = \sqrt{2}$ does not converge to the limit 4 but to the limit 2. This shows that when $a = \sqrt{2}$ the only solution to equation (1.2) that is consistent with the recursive definition of the infinitely continued exponentiation is $x = 2$.

1.2 Explicit difference equations

We now generalize the class of difference equations discussed in the previous section in three different ways. First, we consider systems that are described by a finite-dimensional vector rather than by a single real number; second, we allow the system dynamics f to change in the course of time; and, third, we drop the assumption that the value of the system variable x in a given period depends only on the corresponding value in the immediately preceding period.