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REPRESENTATION THEORY: A COMBINATORIAL VIEWPOINT

This book discusses the representation theory of symmetric groups, the theory of symmetric functions and the polynomial representation theory of general linear groups. The first chapter provides a detailed account of necessary representation-theoretic background. An important highlight of this book is an innovative treatment of the Robinson–Schensted–Knuth correspondence and its dual by extending Viennot's geometric ideas. Another unique feature is an exposition of the relationship between these correspondences, the representation theory of symmetric groups and alternating groups and the theory of symmetric functions. Schur algebras are introduced very naturally as algebras of distributions on general linear groups. The treatment of Schur–Weyl duality reveals the directness and simplicity of Schur's original treatment of the subject. This book is suitable for graduate students, advanced undergraduates and non-specialists with a background in mathematics or physics.

Amritanshu Prasad is a mathematician at The Institute of Mathematical Sciences, Chennai. He obtained his PhD from the University of Chicago, where he worked on automorphic forms and representations of p-adic groups. His current research interests include representation theory, combinatorics, harmonic analysis and number theory. Prasad has extensive experience in teaching mathematics to undergraduate and graduate students in the US, Canada and India. He has been an associate of the Indian Academy of Sciences and was awarded the Young Scientist Medal by the Indian National Science Academy.

Representation Theory

A Combinatorial Viewpoint

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Preface

This book is based on courses taught to graduate students at The Institute of Mathematical Sciences, Chennai, and undergraduates of Chennai Mathematical Institute. It presents important combinatorial ideas that underpin contemporary research in representation theory in their simplest setting: the representation theory of symmetric groups, the theory of symmetric functions and the polynomial representation theory of general linear groups. Readers who have a knowledge of algebra at the level of Artin's book [1] (undergraduate honours level) should find this book quite easy to read. However, Artin's book is not a strict pre-requisite for reading this book. A good understanding of linear algebra and the definitions of groups, rings and modules will suffice.

A Chapterwise Description

The first chapter is a quick introduction to the basic ideas of representation theory leading up to Schur's theory of characters. This theory is developed using an explicit Wedderburn decomposition of the group algebra. The irreducible characters emerge naturally from this decomposition. Readers should try and get through this chapter as quickly as possible; they can always return to it later when needed. Things get more interesting from Chapter 2 onwards.

Chapter 2 focusses on representations that come from group actions on sets. By constructing enough such representations and studying intertwiners between them, the irreducible representations of the first few symmetric groups are classified. A combinatorial criterion for this method to work in general is also deduced.

The combinatorial criterion of Chapter 2 is proved using the Robinson–Schensted–Knuth correspondence in Chapter 3. This correspondence is constructed by generalizing Viennot's light-and-shadows construction of the Robinson–Schensted algorithm. The classification of irreducible representations of S_n by partitions of n along with a proof of Young's rule are the main results of this chapter.

Chapter 4 introduces the sign character of a symmetric group and shows that twisting by the sign character takes the irreducible representation corresponding to a partition to the representation corresponding to its conjugate partition. Young's construction of the irreducible representations of S_n is deduced, and the relationship between these results and the dual RSK correspondence is explained. A light-and-shadows type construction for the dual RSK correspondence is also provided. In the last section of this chapter, the irreducible representations of the alternating groups A_n are classified, and some ideas involved in computing their character tables are outlined with the help of examples. The complete determination of the character table of A_n is postponed to Chapter 5.

Chapter 5 concerns the algebra of symmetric functions. Bases of this algebra consisting of monomial symmetric functions, elementary symmetric functions, complete symmetric functions, power sum symmetric functions and Schur functions are introduced. Combinatorial interpretations of the transition matrices between these bases are provided. The RSK correspondence and its dual are used to understand and organize these transition matrices. Three different formulae for Schur functions are provided: Kostka's combinatorial definition using semistandard Young tableaux, Cauchy's bi-alternant formula and the formulae of Jacobi and Trudi. Frobenius's beautiful formula for characters of a symmetric group using symmetric functions is a highlight of this chapter. This result motivates the definition of Frobenius's characteristic function, which associates symmetric functions to class functions on S_n. Frobenius's characteristic function is used to deduce branching rules for the restriction of representations of S_n to S_{n-1} and to provide a representation-theoretic interpretation of the Littlewood–Richardson coefficients. Combining the characteristic function with the Jacobi–Trudi identity allows for the deduction of the recursive Murnaghan–Nakayama formula, which is a fast algorithm for computing a character value of a symmetric group. With the help of the recursive Murnaghan–Nakayama formula, the character tables of alternating groups are computed.

Chapter 6 treats the polynomial representation theory of general linear groups. Schur algebras are introduced as algebras of homogeneous polynomial distributions on general linear groups. The modules of Schur algebras are shown to correspond to polynomial representations of general linear groups. By interpreting Schur algebras as endomorphism algebras for the actions of symmetric groups on tensor spaces (Schur–Weyl duality), their simple modules are classified. It is shown that polynomial representations of general linear groups are determined by their character values on diagonal matrices or by their restrictions to the subgroup of diagonal matrices (weight spaces). A combinatorial interpretation of the weight space decomposition of a simple polynomial representation of a general linear group is provided.

About the Exercises

Exercises are interspersed with the text throughout this book. Sometimes important steps in proofs are left as exercises. This gives the reader a chance to think about them carefully. In such cases, unless the exercise is very straightforward, at least a sketch of the solution is always provided. Many other exercises also come with solutions. Readers should make multiple attempts to solve an exercise before looking at the solution. Sometimes reading ahead to the end of the chapter or rereading relevant sections may help in solving them.

Exercises are assigned difficulty levels from 0 to 5, indicated in square brackets at the beginning. Roughly speaking the difficulty levels are decided based on the following key:

[0] trivial
[1] routine and almost immediate
[2] follows from a careful understanding of the material presented
[3] a new idea is needed
[4] a clever application of a theorem from the text or elsewhere is needed
[5] needs sustained work with several new ideas

Acknowledgments

This book would not have been possible had it not been for the support and encouragement that I received from my colleagues K. N. Raghavan and S. Viswanath.

Kannappan Sampath read through large parts of this book, discussed them at length with me and suggested many improvements. He also contributed some exercises and many solutions.

Important parts of this book have been influenced by my interactions with my long-time collaborators Uri Onn and Pooja Singla. The idea that the RSK correspondence can be used to classify the simple representations of S_n and prove Young's rule was suggested by their article [23]. Later on, I learned about Schur's purely algebraic approach to what is now known as Schur–Weyl duality from a series of lectures given by Singla based on Green's book [10]. She also suggested Exercise 2.4.6 to me.

My discussions with T. Geetha were instrumental in shaping many sections of this book, particularly the detailed treatment of the representation theory of alternating groups.

Exercise 1.7.17 was suggested by Jan-Christoph Schlage-Puchta. Ananth Shankar suggested the simple proof of Burnside's theorem (Theorem 1.5.17) here. Sudipta Kolay contributed the elegant proof to Part 2 of Lemma 3.1.12. The students of IMSc and CMI who took my courses or read early versions of this book were a constant source of inspiration, as where my PhD students C. P. Anilkumar, Kamlakshya Mahatab and Uday Bhaskar Sharma. Steven Spallone went through a preliminary version of this manuscript and made some helpful suggestions for its improvement. The comments of the anonymous referees have also helped improve this book.

Manish Chaudhary of Cambridge University Press helped me with my book proposal and through the review process. Suvadip Bhattacharjee and Sana Bhanot did an excellent job editing the manuscript.

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