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Basic Concepts of Representation Theory

This chapter contains a fairly self-contained account of the representation theory of finite groups over a field whose characteristic does not divide the order of the group (the semisimple case). The reader who is already familiar with representations, the group algebra, Schur's lemma, characters, and Schur's orthogonality relations could move on to Chapter 2. However, the treatment of these topics in this book may have some new insights for some readers. For instance, the reader will find a careful explanation of why it is that characters (traces of representations) play such an important role in the theory.

1.1 Representations and Modules

Let K be a field and G be a finite group. For a K-vector space V, let $\mathrm{GL}(V)$ denote the group of all invertible K-linear maps $V \to V$.

Definition 1.1.1 (Representation)**.** A representation of G is a pair (ρ, V), where V is a K-vector space and $\rho : G \to \mathrm{GL}(V)$ is a homomorphism of groups.

Definition 1.1.2 (Multiplicative character)**.** A multiplicative character of G is a homomorphism $\chi : G \to K^*$. Here, K^* denotes the multiplicative group of non-zero elements of K.

Example 1.1.3. The simplest example of a multiplicative character $\chi : G \to K^*$ is given by $\chi(g) = 1$ for all $g \in G$. This is called the trivial character of G. A non-trivial character is any character that is different from the trivial character.

Each multiplicative character χ gives rise to a representation as follows: take V to be the one-dimensional vector space K and take ρ to be the homomorphism which takes $g \in G$ to the linear automorphism of K, which multiplies each element by $\chi(g)$. Conversely, every one-dimensional representation comes from a multiplicative character. The representation corresponding to the trivial character of G is called the trivial representation of G.

[1] Exercise 1.1.4. Show that each multiplicative character of G contains $[G, G]$ in its kernel (and therefore descends to a multiplicative character $G/[G, G] \to K^*$). Here, $[G, G]$ denotes the subgroup of G generated by elements of the form $xyx^{-1}y^{-1}$ as x and y run over all elements of G.

[3] Exercise 1.1.5. Let $\chi : G \to K^*$ be a non-trivial multiplicative character. Show that

$$\sum_{g \in G} \chi(g) = 0.$$

Representations of groups can be viewed as modules for certain special types of rings called group algebras. It is assumed that the reader is familiar with the definition of rings, ideals and modules. If not, a quick look at the relevant definitions in a standard textbook (for example, Artin [1, Chapter 12, Section 1]) should suffice.

Definition 1.1.6 (K-algebra)**.** A K-algebra is a ring R whose underlying additive group is a K-vector space and whose multiplication operation $R \times R \to R$ is K-bilinear. Only unital K-algebras will be considered here, namely those with a multiplicative unit.

Example 1.1.7. The space $M_n(K)$ of $n \times n$ matrices with entries in K is a unital K-algebra. If V is an n-dimensional vector space over K, then a choice of basis for V identifies $M_n(K)$ with the algebra $\mathrm{End}_K V$ of K-linear maps $V \to V$.

A left ideal of a K-algebra R is a linear subspace of R which is closed under multiplication on the left by elements of R. Similarly, a right ideal is a linear subspace of R which is closed under multiplication on the right by elements of R. A two-sided ideal is a subspace of R which is both a left and a right ideal.

Example 1.1.8. Let $W \subset K^n$ be a linear subspace. Then

$$\{A \in M_n(K) \mid Ax \in W \text{ for all } x \in K^n\}$$

is a right ideal in $M_n(K)$, while

$$\{A \in M_n(K) \mid Ax = 0 \text{ for all } x \in W\}$$

is a left ideal in $M_n(K)$.

[3] Exercise 1.1.9. Show that $M_n(K)$ has no two-sided ideals except for the two obvious ones, namely $\{0\}$ and $M_n(K)$.

[1] Exercise 1.1.10. Show that if R is a K-algebra and I is a two-sided ideal in R, then the product operation of R descends to a bilinear map $R/I \times R/I \to R/I$ which makes it a K-algebra.

Example 1.1.11. The polynomial algebra $K[x_1, \ldots, x_n]$ is an infinite dimensional commutative unital K-algebra. Every finitely generated[1] commutative K-algebra is a quotient of a polynomial algebra by one of its ideals. The free algebra[2] $K\langle x_1, \ldots, x_n\rangle$ is an infinite dimensional non-commutative algebra. Every finitely generated algebra is a quotient of such an algebra by a two-sided ideal.

A K-algebra homomorphism is a homomorphism of rings, which is also K-linear.

The usual definition of modules for a ring can be adapted to K-algebras:

Definition 1.1.12 (Module)**.** For a K-algebra R, an R-module is a pair $(\tilde{\rho}, V)$, where V is a K-vector space and $\tilde{\rho} : R \to \mathrm{End}_K V$ is a K-algebra homomorphism. We will always assume that $\tilde{\rho}$ maps the unit of R to the unit of $\mathrm{End}_K V$ (such modules are called unital modules).

The notion of an R-module in Definition 1.1.12 requires the K-linearity of $\tilde{\rho}$ and is therefore a little stronger than the general definition of a module for a ring (see, for example, [1, Chapter 12, Section 1]). But the definition above is exactly what is needed to make the correspondence between representations of G and modules of a certain K-algebra $K[G]$ associated to G, as we shall soon see.

Example 1.1.13. Every left ideal of R is an R-module. Any subspace of an R-module M, which is closed under the action of R on M, can be viewed as an R-module in its own right and is called a submodule. A quotient of an R-module by a submodule is also an R-module.

Example 1.1.14. The vector space K^n is an $M_n(K)$-module when vectors in K^n are written as columns and $M_n(K)$ acts by matrix multiplication on the left.

The group algebra $K[G]$ of the group G is a K-algebra whose additive group is the K-vector space with basis

$$\{1_g | g \in G\}$$

and whose product is defined by bilinearly extending

$$1_g 1_h = 1_{gh} \text{ for all } g, h \in G. \tag{1.1}$$

[1] A subset S of an algebra R is said to be a generating set if R is the smallest algebra containing S. An algebra is said to be finitely generated if it has a finite generating subset.

[2] The free algebra $K\langle x_1, \ldots, x_n\rangle$ has as basis words $x_{i_1} x_{i_2} \cdots x_{i_m}$ in the symbols $x_1, x_2, \ldots, x_m$. Basis elements are multiplied by concatenating the corresponding words.

Another useful way of thinking about the group algebra is as the algebra of K-valued functions on G with the product given by convolution: if f_1 and f_2 are two K-valued functions on G, their convolution $f_1 * f_2$ is defined by

$$f_1 * f_2(g) = \sum_{xy=g} f_1(x)f_2(y) \text{ for all } g \in G. \tag{1.2}$$

[2] Exercise 1.1.15. Identify 1_g with the function whose value at g is 1 and which vanishes everywhere else. Under this identification, show that the two definitions of the group algebra given above are equivalent.

[3] Exercise 1.1.16. Let $n > 1$ be an integer. Show that $K[\mathbf{Z}/n\mathbf{Z}]$ is isomorphic to $K[t]/(t^n - 1)$ as an algebra. Here, $(t^n - 1)$ denotes the ideal in $K[t]$ generated by $t^n - 1$.

If $\rho : G \to GL(V)$ is a representation, and one defines a K-algebra homomorphism $\tilde{\rho} : K[G] \to \mathrm{End}_K(V)$ by

$$\tilde{\rho} : f \mapsto \sum_{g\in G} f(g)\rho(g) \tag{1.3}$$

for each $f \in K[G]$, then $(\tilde{\rho}, V)$ is a $K[G]$-module.

Conversely, suppose that $\tilde{\rho} : K[G] \to \mathrm{End}_K(V)$ is a $K[G]$-module. Note that if e denotes the identity element of G, then 1_e is the multiplicative unit of $K[G]$. Since we have assumed that $\tilde{\rho}(1_e) = \mathrm{id}_V$ (such a module is called unital), then for any $g \in G$,

$$\tilde{\rho}(1_g)\tilde{\rho}(1_{g^{-1}}) = \tilde{\rho}(1_e) = \mathrm{id}_V,$$

so $\tilde{\rho}(1_g) \in GL(V)$. Define a representation ρ of G by

$$\rho(g) = \tilde{\rho}(1_g). \tag{1.4}$$

The prescriptions (1.3) and (1.4) define an equivalence between representations of G and unital $K[G]$-modules. This correspondence makes it possible to use concepts from ring theory in the study of group representations.

Example 1.1.17 (Regular representation). For each $r \in R$, define $\tilde{L}(r)$ to be the linear endomorphism of R obtained by left multiplication by r. This turns R into an R-module, which is known as the left regular R-module.

Let us examine the above construction in the case where $R = K[G]$. The group ring $K[G]$ becomes a representation of G if we define $L(g) : K[G] \to K[G]$ by

$$L(g)1_x = \tilde{L}(1_g)1_x = 1_{gx}.$$

This representation is known as the left regular representation of G. If we define $R : G \to GL(K[G])$ by

$$R(g)1_x = 1_{xg^{-1}},$$

we get another representation of G on $K[G]$, which is known as the right regular representation of G.

[1] Exercise 1.1.18. If $K[G]$ is viewed as the space of K-valued functions on G (as in Exercise 1.1.15), then

$$(L(g)f)(x) = f(g^{-1}x) \text{ and } (R(g)f)(x) = f(xg).$$

1.2 Invariant Subspaces and Simplicity

Definition 1.2.1 (Invariant subspace)**.** A subspace W of V is called an invariant subspace for a representation $\rho : G \to \mathrm{GL}(V)$ if $\rho(g)W \subset W$ for all $g \in G$.

Similarly, a subspace W of V is called an invariant subspace for an R-module $\tilde{\rho} : R \to \mathrm{End}_K V$ if $\tilde{\rho}(r)W \subset W$ for all $r \in R$. [3]

Example 1.2.2. For the left regular representation $(L, K[G])$, the subspace of constant functions is a one-dimensional invariant subspace. The subspace

$$K[G]_0 = \Big\{ f : G \to K \mid \sum_{g \in G} f(g) = 0 \Big\}$$

is an invariant subspace of dimension $|G| - 1$.

[3] Exercise 1.2.3. The subspace $K[G]_0$ has an invariant complement in $(L, K[G])$ if and only if $|G|$ is not divisible by the characteristic of K (this includes the case where K has characteristic zero).

[1] Exercise 1.2.4. Let $G = \mathbf{Z}/2\mathbf{Z}$ and let K be a field of characteristic two. Show that the subspace of $K[G]$ spanned by $1_0 + 1_1$ is the only non-trivial proper invariant subspace for the left (or right) regular representation of G.

[3] Exercise 1.2.5. Show that if every representation of a group is a sum of one-dimensional invariant subspaces, then the group is abelian. Hint: Use Exercise 1.1.4 and the regular representation.

Definition 1.2.6 (Simplicity)**.** A representation or module is said to be simple (or irreducible) if it has no non-trivial proper invariant subspaces. As a convention, the representation or module of dimension zero is not considered to be simple.[4]

Example 1.2.7. Every one-dimensional representation is simple.

[3] Exercise 1.2.8. Every simple module for a finite-dimensional K-algebra is finite dimensional.

[3] An invariant subspace of a representation is often called a *subrepresentation*, and an invariant subspace of a module is usually called a *submodule*.

[4] This is a little bit like 1 not being considered a prime number.

[5] Exercise 1.2.9. If K is algebraically closed, and G is abelian, then every simple representation of G is of dimension one. Hint: Show that for any commuting family of matrices in an algebraically closed field, there is a basis with respect to which all the matrices in that family are upper triangular.[5]

Example 1.2.10. The hypothesis that K is algebraically closed is necessary in Exercise 1.2.9. Take for example, $G = \mathbf{Z}/4\mathbf{Z}$ and $\rho : G \to GL_2(\mathbf{R})$ the representation which takes a generator of $\mathbf{Z}/4\mathbf{Z}$ to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since this matrix is a rotation by $\pi/2$, no line in $\mathbf{R}^2$ is left invariant by it, and so the abelian group $\mathbf{Z}/4\mathbf{Z}$ admits a simple two-dimensional representation over real numbers.

Definition 1.2.11 (Intertwiners)**.** Let (ρ_1, V_1) and (ρ_2, V_2) be representations of G. A linear transformation $T : V_1 \to V_2$ is called an intertwiner (or a G-homomorphism) if

$$T \circ \rho_1(g) = \rho_2(g) \circ T \text{ for all } g \in G. \tag{1.5}$$

The space of all intertwiners $V_1 \to V_2$ is denoted $\mathrm{Hom}_G(V_1, V_2)$.

Similarly, for R-modules $(\tilde{\rho}_1, V_1)$ and $(\tilde{\rho}_2, V_2)$, an intertwiner is a linear transformation $T : V_1 \to V_2$ such that

$$T \circ \tilde{\rho}_1(r) = \tilde{\rho}_2(r) \circ T \text{ for all } r \in R.$$

The space of all such intertwiners is denoted by $\mathrm{Hom}_R(V_1, V_2)$.

The intertwiner condition (1.5) can be visualized as a commutative diagram:

$$\begin{array}{ccc} V_1 & \xrightarrow{T} & V_2 \\ {\scriptstyle \rho_1(g)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \rho_2(g)} \\ V_1 & \xrightarrow[T]{} & V_2 \end{array}$$

If one begins with an element in the top-right corner of this diagram, the images obtained by applying the functions along either of the two paths to the bottom-right corner are the same.

[1] Exercise 1.2.12. The kernel of an intertwiner is an invariant subspace of its domain, and the image is an invariant subspace of its codomain.

Theorem 1.2.13 (Schur's lemma I)**.** *If K is algebraically closed and V is a finite-dimensional simple representation of G, then every self-intertwiner $T : V \to V$ is a scalar multiple of the identity map. In other words,* $\mathrm{End}_G V = K\mathrm{id}_V$ *(*$\mathrm{End}_G V$ *denotes* $\mathrm{Hom}_G(V, V)$*, the self-intertwiners of V, which are also called G-endomorphisms of V).*

5 Exercise 1.2.9 becomes much easier if Schur's lemma (Theorem 1.2.13) is used instead of the hint.

Proof. Since K is algebraically closed, any self-intertwiner $T : V \to V$ has an eigenvalue, say λ. Now $T - \lambda \mathrm{id}_V$ is also an intertwiner. Moreover, it has a non-trivial kernel. Since its kernel is an invariant subspace (Exercise 1.2.12), it must (by the simplicity of V) be all of V. Therefore, $T = \lambda \mathrm{id}_V$. □

A similar statement (with the same proof) holds for simple modules of a K-algebra.

[1] Exercise 1.2.14 (Central character). When K is algebraically closed, show that the centre $Z(G)$ of G acts on any simple representation by scalar matrices (if $g \in Z(G)$ acts by the scalar matrix $\lambda(g)I$, then $g \mapsto \lambda(g)$ is a homomorphism $Z(G) \to K^*$, which is called the central character of the representation).

[1] Exercise 1.2.15 (Schur's lemma for arbitrary fields). Let K be any field (not necessarily algebraically closed). Show that any non-zero self-intertwiner of a simple representation (or module) is invertible.

Definition 1.2.16 (Isomorphism). We say that representations (or modules) V_1 and V_2 are isomorphic (and write $V_1 \cong V_2$ or $\rho_1 \cong \rho_2$) if there exists an invertible intertwiner $V_1 \to V_2$ (its inverse will be an intertwiner $V_2 \to V_1$).

Theorem 1.2.17 (Schur's lemma II). *If V_1 and V_2 are simple, then every non-zero intertwiner $T : V_1 \to V_2$ is an isomorphism. Consequently, either $V_1 \cong V_2$ or there are no non-zero intertwiners $V_1 \to V_2$.*

Proof. If T is a non-zero intertwiner, then its kernel is an invariant subspace of V_1. Since this kernel cannot be all of V_1, it is trivial; hence, T is injective. Its image, being a non-trivial invariant subspace of V_2, must be all of V_2; therefore, T is an isomorphism. □

An easy consequence of the two previous results is

Corollary 1.2.18. *If K is algebraically closed, V_1 and V_2 are simple and $T : V_1 \to V_2$ is any non-trivial intertwiner, then* $\mathrm{Hom}_G(V_1, V_2) = KT$.

Proof. T is invertible by Schur's Lemma II. If $S : V_1 \to V_2$ is another intertwiner, then $T^{-1} \circ S$ is a self-intertwiner of V_1. By Schur's Lemma I, $T^{-1}S = \lambda \mathrm{id}_{V_1}$ for some $\lambda \in K$, whence $S = \lambda T$. □

1.3 Complete Reducibility

Definition 1.3.1 (Completely reducible module). An R-module is said to be completely reducible if it is a direct sum of simple modules.

We have already seen (Exercises 1.2.3 and 1.2.4) that not all modules are completely reducible. From now on, in order to not get distracted by issues that are interesting, but ultimately incidental to the subject matter of this book, we will only consider finite-dimensional K-algebras and their finite-dimensional modules until Chapter 6.

[2] Exercise 1.3.2. Assume that every invariant subspace of an R-module V admits an invariant complement. Let W be an invariant subspace of V. Show that every invariant subspace of W admits an invariant complement in W and that every invariant subspace of V/W admits an invariant complement in V/W.

[3] Exercise 1.3.3. Show that an R-module is completely reducible if and only if every invariant subspace has an invariant complement.

[3] Exercise 1.3.4. Show that if the left regular R-module is completely reducible, then every R-module is completely reducible.

If V is a finite-dimensional completely reducible R-module, then

$$V \cong V_1^{\oplus m_1} \oplus V_2^{\oplus m_2} \oplus \cdots \oplus V_r^{\oplus m_r}, \tag{1.6}$$

where (by grouping the simple subspaces of V which are isomorphic together) $V_1, V_2, \ldots, V_r$ is a collection of pairwise non-isomorphic simple R-modules. The number m_k is called the multiplicity of V_k in V. We shall refer to (1.6) as the decomposition of V into simple modules with multiplicities. Let W be another finite-dimensional completely reducible module whose decomposition into simple modules with multiplicities is

$$W \cong V_1^{\oplus n_1} \oplus V_2^{\oplus n_2} \oplus \cdots V_r^{\oplus n_r} \tag{1.7}$$

(by allowing some of the n_k's and m_k's to be 0, we may assume that the underlying collection $V_1, V_2, \ldots, V_r$ of simple modules is the same for V and W). Since there are no intertwiners $V_i \to V_j$ for $i \neq j$, any $T \in \mathrm{Hom}_R(W, V)$ can be expressed as

$$T = \bigoplus_k T_k,$$

where $T_k : V_k^{\oplus n_k} \to V_k^{\oplus m_k}$ is an intertwiner. Represent an element $x \in V_k^{\oplus n_k}$ as a vector $(x_1, \ldots, x_{n_k})$ and $y \in V_k^{\oplus m_k}$ as $y = (y_1, \ldots, y_{m_k})$, with each $x_i, y_i \in V_k$. Writing these vectors as columns, the intertwiner T_k can itself be expressed as an $m_k \times n_k$ matrix $T_k = (T_{ij})$ (where $T_{ij} \in \mathrm{End}_R V_k$) using

$$\begin{pmatrix} T(x)_1 \\ T(x)_2 \\ \vdots \\ T(x)_{m_k} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n_k} \\ T_{21} & T_{22} & \cdots & T_{2n_k} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m_k1} & T_{m_k2} & \cdots & T_{m_kn_k} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_k} \end{pmatrix}.$$

Thus, the entries of the matrix, being scalar multiples of the identity id_{V_k}, can themselves be thought of as scalars, allowing us to write

$$\mathrm{Hom}_R(W, V) = \bigoplus_{k=1}^{r} M_{m_k \times n_k}(K),$$

where $M_{m_k \times n_k}$ denotes the set of $m_k \times n_k$ matrices with entries in K. One easily checks that composition of intertwiners expressed as matrices in the above manner corresponds to multiplication of matrices.

Theorem 1.3.5. *If K is algebraically closed and V and W have decompositions into sums of simple modules with multiplicities given by (1.6) and (1.7), then*

$$\dim \mathrm{Hom}_R(V, W) = \dim \mathrm{Hom}_R(W, V) = \sum_i m_i n_i.$$

In the special case where $W = V$, we obtain

Theorem 1.3.6. *Let K be an algebraically closed field and R be a K-algebra. If the R-module V is a sum of non-isomorphic simple modules with multiplicities given by* (1.6)*, then* $\mathrm{End}_R V$ *is a sum of matrix algebras (with componentwise multiplication):*

$$\mathrm{End}_R V \cong \bigoplus_{i=1}^{r} M_{m_i}(K),$$

where the right-hand side should be interpreted as a sum of algebras.

Sum of algebras

The notion of a sum of algebras will come up often and therefore deserves a short discussion.

Definition 1.3.7 (Sum of algebras)**.** If $R_1, R_2, \ldots, R_k$ are algebras, their sum is the algebra whose underlying vector space is the direct sum $R := R_1 \oplus R_2 \oplus \cdots \oplus R_k$, with multiplication defined componentwise:

$$(r_1 + r_2 + \cdots + r_k)(s_1 + s_2 + \cdots + s_k) = r_1 s_1 + r_2 s_2 + \cdots + r_k s_k$$

Thus, each R_i is a subalgebra of R for each i. If each of the algebras R_i is unital with unit 1_i, then the sum

$$1 := 1_1 + 1_2 + \cdots + 1_k$$

is the multiplicative unit for R. In particular, R is also unital. If $(\tilde{\rho}_i, M_i)$ is a unital R_i module (meaning that $\tilde{\rho}_i(1_i) = \mathrm{id}_{M_i}$), then

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$$

is also a unital R-module when $\tilde{\rho} : R \to \mathrm{End}_K M$ is defined by

$$\tilde{\rho}(r_1 + r_2 + \cdots + r_k) := \tilde{\rho}_1(r_1) + \tilde{\rho}_2(r_2) + \cdots + \tilde{\rho}_k(r_k).$$

The M_i's can be recovered from M by $M_i = \tilde{\rho}(1_i)M$. Thus, R-modules correspond precisely to collections of R_i-modules (one for each i).

On a purely combinatorial level

Theorem 1.3.8. *Assume that K is algebraically closed. If the R-module V is a sum of non-isomorphic simple modules with multiplicities given by* (1.6)*, then*

$$\dim \mathrm{End}_R V = \sum_{i=1}^{r} m_i^2.$$

Recall that the centre of a K-algebra R consists of those elements which commute with every element of R.

We all know that the centre of a matrix algebra consists of scalar matrices. The centre of a direct sum of algebras is the direct sum of their centres. It follows that the dimension of the centre of $\bigoplus_{i=1}^{r} M_{m_i}(K)$ is the number of i such that $m_i > 0$. Thus, a consequence of Theorem 1.3.6 is

Theorem 1.3.9. *Let R be a K-algebra, with K algebraically closed. If the R-module V is a sum of non-isomorphic simple modules with multiplicities given by* (1.6) *with all the multiplicities $m_i > 0$, then the dimension of the centre of* $\mathrm{End}_R V$ *is r.*

The next exercise is a trivial consequence of Theorem 1.3.8

[0] Exercise 1.3.10. Let R be a K-algebra, where K is an algebraically closed field. Show that a completely reducible R-module V is simple if and only if

$$\dim \mathrm{End}_R V = 1.$$

And similarly, Theorem 1.3.5 can be used to solve the following:

[0] Exercise 1.3.11. Assume that K is algebraically closed, V is simple and W is completely reducible. Then, $\dim \mathrm{Hom}_R(V, W)$ is the multiplicity of V in W.

For the following exercise, use Theorem 1.3.6

[1] Exercise 1.3.12. Assume that K is algebraically closed. A completely reducible R-module V has a multiplicity-free decomposition (meaning that its decomposition into simple modules with multiplicities is of the form (1.6) with $m_i = 1$ for all i) if and only if its endomorphism algebra $\mathrm{End}_R V$ is commutative.